Chapter 4: Discrete Probability Models

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1 Introduction

In the previous chapter we learned about how to describe the distributions and their summary measures of random variables and random vectors in general, which included both discrete and continuous cases. In this chapter we shall study some specific discrete distributions, which frequently arise in practice. The reason for doing so is that, many specific situations can be modeled using one of these distributions, and thus learning these discrete probability models in abstract terms helps one to immediately apply the general results pertaining to these distributions to special cases. We begin our discussion by defining a fundamental concept, called **Bernoulli trial**, which is the basis of most of these discrete distributions.

Definition 4.1: A chance experiment which has only two possible outcomes, called Success(S) and Failure(F), is called a **Bernoulli trial**, and the probability of Success in such a trial is denoted by $P(S) = p$, so that $P(F) = 1 - p = q$ (say).

Basically a Bernoulli trial is just an abstraction of the coin tossing experiment, in which there are only two possible outcomes - Head and Tail. A large number of practical situations can be modeled as a Bernoulli trial. For example if you are interested in a certain brand and observe the brand preference of a consumer, the result can be coded as a Success if the consumer prefers that brand and Failure otherwise. At the end of the trading day the closing price of a stock or a market index might go up (Success) or come down (Failure). A given employee on a given day might be absent (Success) or present (Failure). A candidate at the end of an interview is either selected (Success) or rejected (Failure). To a customer walking into a car dealership a particular sales person either is able to sale a car (Success) or cannot (Failure). At the end of the term a corporate bond either defaults (Success) or is able to fulfill its obligations (Failure). In a warehouse, a truck full of inventories is either unloaded within an hour (Success) or it takes more than an hour (Failure). Thus we see that in a large number of practical situations the outcome of a chance experiment is dichotomous and is thus amenable to be modeled as a Bernoulli trial, which is the basic building block of most of the discrete models discussed in this chapter.

Discussion of each of these discrete probability models will follow the following common pattern. First the random variable will be defined non-mathematically in words, so that one can readily apply it in a specific practical situation. Next the p.m.f. of the random variable (and in rare cases where closed form expression for c.d.f. exists) will be derived from this verbal definition. Next the p.m.f. will be studied analytically with an eye towards its visualization. After that the key summary measures, in particular, the first two moments, will be derived for each of these distributions. Finally, if the distribution posses any special property, that is proved in the end. A few examples involving each distribution are also worked out at the end of each section.

2 Binomial Distribution

Consider n independent and identically distributed, called i.i.d., Bernoulli trials. By independent it is meant that $P(S/F$ in the i-th trial | outcomes of any other trial(s)) $= P(S/F$ in the i-th trial) and by identical it is meant that the probability of Success p of the Bernoulli trial does not change from trial to trial.

Definition 4.2: The random variable X denoting the total number of Successes in n i.i.d. Bernoulli trials is said to have a **Binomial Distribution** and is denoted by $X \sim \mathrm{B}(n,p)$.

Thus for example, if the probability of a dentist recommending toothpaste Brand X is 0.3, in a random sample of 10 dentists, the number of dentists recommending toothpaste Brand X is a B(10,0.3) random variable. If 80% of small scale manufacturing units have a debt-to-equity ratio more than 1, in a random sample of 25 such units, the number of units having debt-to-equity ratio more than 1 is a B(25,0.8) random variable. If the probability of a mouse developing a tumor subjected to a certain carcinogen is 0.4, the number of mice developing tumor in a laboratory experiment where 8 such mice were subjected to that carcinogen is a B(8,0.4) random variable.

2.1 P.m.f. & C.d.f.

Let $X \sim \mathrm{B}(n,p)$. Since X denotes the number of Successes in n trials, it must be an integer between 0 and n, and thus $\mathcal{X}$, the set of possible values that X can take equals $\{0,1,\ldots,n\}$. This is the first step towards specifying the p.m.f. of X. Next fix an $x \in \mathcal{X}$ and we are now interested in figuring out $p(x) = P[X = x]$, the p.m.f. of X.

First let us count the total number of outcomes that is possible for this experiment of conducting n i.i.d. Bernoulli trials. A typical outcome may be represented by a string of S's and F's of length n which may be expressed as $\underbrace{FS\ldots\ldots FF}_{n-\text{many}}$. Thus the total number of possible outcomes is same as the total number of possible strings of S's and F's of length n. Since there are n positions in this string and for each position we have two choices (an S or an F) to fill it up, the total possible number of such strings equal 2^n.

Now note that all these strings are <u>not</u> equally likely, as in general $P(S) = p \neq \frac{1}{2}$. Probability of an arbitrary string like $FS\ldots FF$ depends on the number of S's and F's it has in it. A typical outcome (represented by a string of S's and F's of length n) belonging to the event of interest $[X = x]$ must be such that it has exactly x-many S's in it so that it automatically has $(n-x)$-many F's. This is because the event $[X = x]$ says that the total number of Successes in the n trials equal x. Thus the probability of a typical outcome belonging to the event $[X = x]$ is given by

$$P\left[\underbrace{FS\ldots\ldots\ldots FF}_{x-S\text{'s } \& \ (n-x)-F\text{'s}}\right] = \underbrace{P(F)P(S)\ldots\ldots P(F)P(F)}_{x-P(S)\text{'s } \& \ (n-x)-P(F)\text{'s}} = [P(S)]^x\,[P(F)]^{n-x} = p^x q^{n-x}.$$

The first equality follows from the independence assumption and the next equality follows from the identical distribution assumption. Thus all the basic outcomes belonging to the event $[X = x]$ have the same probability $p^x q^{n-x}$. of occurrence, although all the 2^n possible outcomes do not have equal probability. Hence $P[X = x]$ is easily computed once we can figure out the the total number of outcomes in the event $[X = x]$.

Total number of outcomes in the event $[X = x]$ is same as the number of strings of S's and F's of length n containing exactly x-many S's and $(n - x)$-many F's. This is same as the number of ways one can choose the x-many positions for the S's in this string out of the positions $1, \ldots, n$. Then after choosing these x-many positions, just label them as S's and label the remaining $(n - x)$-many positions by F's. The number of ways once can choose x positions out of a total possible n is given by $\begin{pmatrix} n \\ x \end{pmatrix}$ and thus there are as many outcomes in the event $[X = x]$. Hence for

$$X \sim \mathrm{B}(n, p), \qquad P[X = x] = \begin{pmatrix} n \\ x \end{pmatrix} p^x q^{n-x} \text{ for } x = 0, 1, \ldots, n \tag{1}$$

Once the formula for the p.m.f. is obtained, the next task is to attempt to figure out the c.d.f. in closed form, if possible. Unfortunately for the $\mathrm{B}(n, p)$ model, there is no closed form expression for the c.d.f. and the only way to compute it is by brute force summation. Recall that the c.d.f. is required for probability calculations, for which the p.m.f. is rather unwieldy. In any case for the sake of completeness, the expression for the Binomial c.d.f. is given below:

$$\text{For } X \sim \mathrm{B}(n, p), \qquad F(x) = P[X \leq x] = \begin{cases} 0 & \text{if } x < 0 \\ \sum_{k=0}^{\lfloor x \rfloor} \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} & \text{if } 0 \leq x < n \\ 1 & \text{if } x \geq n \end{cases}.$$

Here $\lfloor x \rfloor$ is the integer part of x, which is same as the greatest integer $\leq x$. C.d.f.'s of $\mathrm{B}(n, p)$ distributions, for $n = 2, 3, \ldots, 30$ and $p = 0.05, 0.10, \ldots, 0.50$ have been tabulated in Table I in the Appendix of this chapter. For other values of $p \in (0, 0.5)$ use interpolation, and for $p \in (0.5, 1)$ use the fact that if $X \sim \mathrm{B}(n, p)$, then $n - X \sim \mathrm{B}(n, q)$, recast the problem in terms $n - X$ and then use Table I. A couple of examples should help clarify these steps.

Example 4.1: A student has to take a multiple choice Statistics pop-quiz with no preparation. The quiz has 20 questions and each question has four choices A, B, C and D as its answer. Since the student is unprepared, he randomly guesses the answers of each question independently of each other. What is the probability that he gets at least 10 correct answer?
Solution: Answering each question may be construed as a Bernoulli trial, where Success indicates getting the answer correct as a result of the random guess. Since there are four choices to each question, probability of Success of this Bernoulli trial is 0.25, and this success probability does not change from question to question or trial to trial. Furthermore the student guesses the answer for each of the 20 questions independently of each other. Thus we have a sequence of 20 i.i.d. Bernoulli trials with $P(S) = 0.25$, and we are concerned with the total number of Successes, which is the total number of correct answers. Thus if

we denote this total number of correct answers by X, we have $X \sim \text{B}(20, 0.25)$ and we are interested in finding $P[X \geq 10]$.

$$P[X \geq 10] = 1 - P[X < 10] = 1 - P[X \leq 9] = 1 - F(9) = 1 - 0.9861 = 0.0139$$

where $F(9) = 0.9861$ is directly read from Table I under $n = 20$, $p = 0.25$ and $x = 9$. Thus the student has a very little chance (1.39%) of getting at least half of the answers right by pure random guess. $\bigtriangledown$

Example 4.2: The market share of toothpaste brand A is 32%. What is the probability that in a random sample of 22 people exactly 7 (≈32% of 22) people say that they are user of the toothpaste brand A? Assume that the toothpaste brand choice of the sampled people are independent of each other.

Solution: For every person in the sample say it is a Success if s/he is a user of toothpaste brand A and Failure otherwise. Since the market share of toothpaste brand A is 32%, the probability of Success is 0.32. Thus here again we have a sequence of 22 i.i.d. Bernoulli trials with a success probability of 0.32 and we are concerned with the total number of Successes in these 22 trials. If we denote the number of people (in the sample of 22) using toothpaste brand A by X, then we have $X \sim \text{B}(22, 0.32)$ and we are interested in $P[X = 7]$. Note that $P[X = 7] = P[X \leq 7 - P[X \leq 6] = F(7) - F(6)$. However unfortunately the c.d.f. $F(\cdot)$ of the $\text{B}(22, 0.32)$ distribution is not available in Table I. Nevertheless one can still use linear interpolation and two closest values to get an approximate value of $F(7)$ and $F(6)$ of the $\text{B}(22, 0.32)$ distribution. The two closest values are obtained by looking at the $\text{B}(22, 0.30)$ and $\text{B}(22, 0.35)$ distributions. $F_{B(22,0.30)}(6) = 0.4942$, $F_{B(22,0.35)}(6) = 0.3022$, $F_{B(22,0.30)}(7) = 0.6713$, and $F_{B(22,0.35)}(7) = 0.4736$. Thus by linear interpolation

$$F_{B(22,0.32)}(6) \approx 0.3022 + (0.35 - 0.32) \times \frac{0.4942 - 0.3022}{0.35 - 0.30} = 0.4174$$

and

$$F_{B(22,0.32)}(7) \approx 0.4736 + (0.35 - 0.32) \times \frac{0.6713 - 0.4736}{0.35 - 0.30} = 0.5922.$$

Therefore $P[X = 7] \approx 0.5922 - 0.4174 = 0.1748$. Note that for this problem, obviously it is much easier to figure out $P[X = 7]$ directly using the binomial p.m.f. formula given in (1), which gives

$$P[X = 7] = \binom{22}{7}(0.32)^7(0.68)^{15} = 0.1801.$$

However the purpose of this problem was to illustrate how one can use linear interpolation to figure out c.d.f. values for other binomial distributions not tabulated in Table I. $\bigtriangledown$

Example 4.3: Maitrayee, the topper of the graduating MBA batch, assesses that the probability of her clearing a job interview is 0.8. She appears for 5 such interviews. Assume that her performances in the interviews are independent of each other. What is the probability that she lands up with at least two job offers?

Solution: Think of each interview as a Bernoulli trial and its outcome a Success if Maitrayee clears it and Failure otherwise. Thus we have 5 i.i.d. Bernoulli trials and we are concerned with total number of Successes in these trials. Thus if we denote the number of offers received by Maitrayee by X, we have $X \sim$B(5,0.8) and we are interested in $P[X \geq 2]$.

$$P[X \geq 2] = P[-X \leq -2] = P[5 - X \leq 3] = 0.9933.$$

The value 0.9933 is directly read from Table I under $n = 5$, $p = 0.20$ and $x = 3$. This is because since $X \sim$B(20,0.8), $5 - X \sim$B(5,0.2). $\bigtriangledown$

Calculation of Binomial probabilities for $n > 30$ requires an approximation method which will be discussed in the next chapter, under the section on Normal distribution and is hence deferred till then.

So far we have just derived the p.m.f. of the Binomial distribution and illustrated how to compute probabilities involving a Binomial distribution using the tabulated c.d.f.. But these are not all. Along with these we need to have a mental picture of how the distribution looks like in general, and formulæ for its moments and quantiles, in the same spirit of studying any arbitrary discrete distribution, as was done in §2 of the previous chapter. Thus we take up these issues one by one starting with the task of building a mental picture of the distribution.

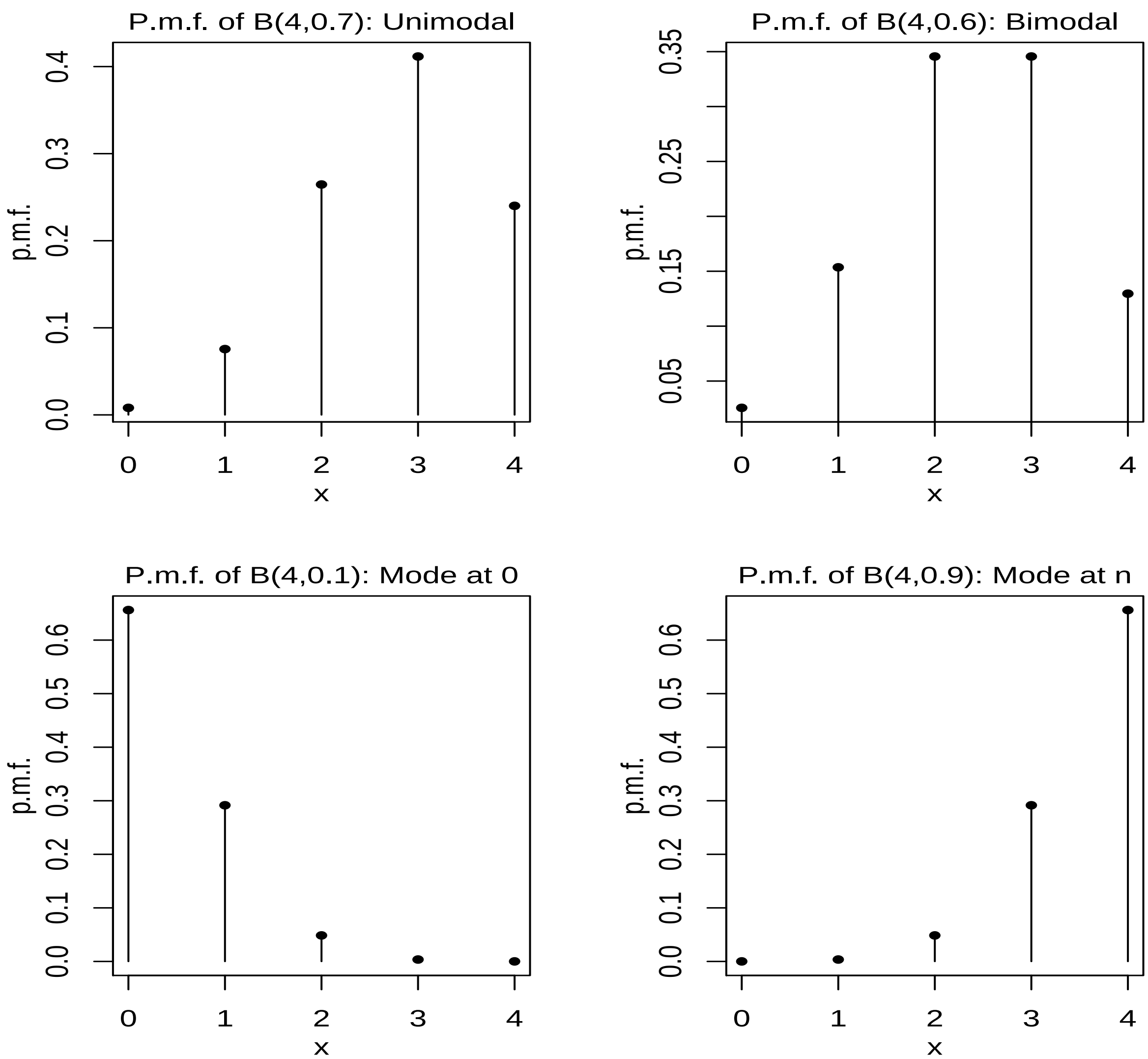
2.2 Picturing the Distribution

Consider two successive terms $P[X = x]$ and $P[X = x + 1]$ of the Binomial p.m.f. given in (1).

$$\begin{aligned}
& \frac{P[X = x]}{P[X = x+1]} && \substack{< \\ = \\ >} \quad 1 \\
\iff \quad & \frac{n!(x+1)!(n-x-1)!}{n!x!(n-x)!} \frac{p^x q^{n-x}}{p^{x+1} q^{n-x-1}} && \substack{< \\ = \\ >} \quad 1 \\
\iff \quad & \frac{x+1}{n-x} \frac{q}{p} && \substack{< \\ = \\ >} \quad 1 \\
\iff \quad & x + 1 - px - p && \substack{< \\ = \\ >} \quad np - px \\
\iff \quad & x + 1 && \substack{< \\ = \\ >} \quad (n+1)p
\end{aligned}$$

Thus the Binomial p.m.f. at $(x+1)$ is greater than that at the previous point x as long as it *i.e.* $(x+1)$ is less than $(n+1)p$, and then when it exceeds $(n+1)p$, $P[X = x+1] < P[X = x]$. If $(n+1)p$ is an integer, then the binomial p.m.f. is equal at both $(n+1)p$ and $(n+1)p - 1$. Thus if $(n+1)p$ is not an integer, the Binomial p.m.f. first increases as x runs through $0, 1, \ldots, \lfloor (n+1)p \rfloor$ and then starts decreasing as x runs through $\lfloor (n+1)p \rfloor + 1, \lfloor (n+1)p \rfloor + 2, \ldots, n$. On the other hand if $(n+1)p$ is an integer, the Binomial p.m.f. first increases as x runs through $0, 1, \ldots, (n+1)p - 1$, then at $(n+1)p$ it equals that at $(n+1)p - 1$ and then it starts decreasing as as x runs through $\lfloor(n+1)p\rfloor+1, \lfloor(n+1)p\rfloor+2, \ldots, n$. Thus most of the Binomial distributions are unimodal with its mode at $\lfloor(n+1)p\rfloor$. In some rare cases where $(n+1)p$ is an integer, it is bimodal with its two modes lying side by side at $(n+1)p-1$ and $(n+1)p$.

Though the above description conjures up a picture of a hump shaped p.m.f. with a single (or at most two adjacent) mode(s) lying (side by side) somewhere in between 0 and n, a little thought would reveal that this need not necessarily be the case. If p is too small, in particular if $p < \frac{1}{n+1}$, $\lfloor(n+1)p\rfloor = 0$ and thus in this case the mode occurs at 0 and the Binomial p.m.f. is a strictly decreasing function of its argument x. On the other hand, if p is too large, in particular if $p > \frac{n}{n+1}$, $\lfloor(n+1)p\rfloor = n$ and thus in this case the mode occurs at n and the Binomial p.m.f. is a strictly increasing function of its argument x. These four possibilities are depicted in the following plots.



This more or less completes depicting how a Binomial distribution might look like in general. Next we launch our task of providing the key summary measures of the distribution in terms of its moments and quantiles.

2.3 Moments

Although we are learning the properties of the Binomial distribution in detail in this section, recall that this distribution was nonetheless introduced in **Example 25** of the previous chapter, where we had computed its p.g.f. $g(t) = (q + pt)^n$, and using which we had derived its p.m.f., mean and variance[1]. Since the m.g.f. of a non-negative integer valued r.v. is easily derived from its p.g.f. as $M(t) = g(e^t)$, the m.g.f. of the $\mathrm{B}(n, p)$ distribution is given by $M(t) = (q + pe^t)^n$. In **Example 25** of the previous chapter we had used the p.g.f. for computing the Binomial moments. One can also do the same using its m.g.f. as follows:

$$M'(t) = \frac{d}{dt}(q + pe^t)^n = npe^t(q + pe^t)^{n-1} \Rightarrow E[X] = M'(0) = np$$

$$M''(t) = np\frac{d}{dt}\left\{e^t(q + pe^t)^{n-1}\right\} = npe^t(q+pe^t)^{n-2}\left(q + npe^t\right) \Rightarrow E\left[X^2\right] = M''(0) = npq+n^2p^2$$

and thus $V[X] = npq$. Note that the same formulæ were obtained in **Example 25** of the previous chapter. Although we have obtained these formulæ mechanically using the tools of generating functions, it does not provide much insight as to why these are so. Towards this end we shall first demonstrate how to obtain the same formulæ directly using the definition of moments and the Binomial p.m.f..

$$\begin{aligned}
&E[X] \\
&= \sum_{x=0}^{n} xP[X = x] \\
&= \sum_{x=1}^{n} x\frac{n!}{x!(n-x)!}p^x q^{n-x} && \text{(using the Binomial p.m.f. formula (1) and using} \\
& && \text{the fact that the summand is 0 for } x = 0) \\
&= np\sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\left(\overline{n-1} - \overline{x-1}\right)!}p^{x-1}q^{(n-1)-(x-1)} \\
&= np\sum_{y=0}^{n-1}\binom{n-1}{y}p^y q^{(n-1)-y} && \text{(making the change of variable } y = x-1) \\
&= np(p+q)^{n-1} && \text{(by Binomial Theorem)} \\
&= np && \text{(as } p+q = 1)
\end{aligned}$$

$$\begin{aligned}
&E[X(X-1)] \\
&= \sum_{x=0}^{n} x(x-1)P[X = x] && \text{(by the law of unconscious statistician)} \\
&= \sum_{x=2}^{n} x(x-1)\frac{n!}{x!(n-x)!}p^x q^{n-x} && \text{(using the Binomial p.m.f. formula (1) and using}
\end{aligned}$$

[1] Studying a distribution is not complete without studying its generating functions. Since we have already studied these in detail in **Example 25** of the previous chapter, the reader at this point should go back and re-look at **Example 25** of the previous chapter, treating it as a discussion on generating function of the Binomial distribution.

the fact that the summand is 0 for $x = 0$ and 1)

$$
\begin{aligned}
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!\left(\overline{n-2} - \overline{x-2}\right)!} p^{x-2} q^{(n-2)-(x-2)} \\
&= n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y q^{(n-2)-y} \qquad \text{(making the change of variable } y = x-2\text{)} \\
&= n(n-1)p^2 (p+q)^{n-2} \qquad \text{(by Binomial Theorem)} \\
&= n(n-1)p^2 \qquad \text{(as } p+q=1\text{)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& E\left[X^2\right] = E[X(X-1)] + E[X] = n(n-1)p^2 + np = n^2p^2 + np - np^2 \\
\Rightarrow\ & V[X] = n^2p^2 + np - np^2 - n^2p^2 = np(1-p) = npq.
\end{aligned}
$$

Since each trial has success probability of p, in the long run when the trial is repeated over and over again, one would "expect" the proportion of successes to be p. Thus it is intuitively quite reasonable that for n trials the "expected" number of successes are np. As for the variance, not much intuition can be provided for the exact formula, but it is clear that the variance is going to be small if p is either very small (making q large) or very large (making q small). This is because in these cases most of the probability mass of the Binomial random variable is going to be concentrated around 0 and n respectively. Thus intuitively the variance formula should be such that it goes to 0 as $p \to 0$ or $p \to 1$, and has large values at the middle values of p. Also it should be an increasing function of n, because as n increases there is simply more possibilities of values for the Binomial random variable and thus inflating its variance. The expression npq or $np(1-p)$ is possibly the simplest function of n and p which enjoys all these properties. It increases linearly as n increases, and the function $p(1-p)$ is a symmetric parabola on $(0,1)$ with a maxima at $\frac{1}{2}$ and vanishing at the two end-points.

So far the derivation of the mean and variance of the Binomial random variable has relied either on generating function tools or brute-force computation. There is however a very simple derivation, which does not depend on the learner's mathematical prowess. Consider the way the Binomial random variable was introduced in **Example 25** of the previous chapter. For $i = 1, \ldots, n$, for the i-th Bernoulli trial, define $X_i = \begin{cases} 1 & \text{if the } i\text{-th trial is a Success} \\ 0 & \text{if the } i\text{-th trial is a Failure} \end{cases}$. Thus for each $i = 1, \ldots, n$, X_i is a random variable which takes the value 1 with probability p and takes the value 0 with probability q. Furthermore $X_1, \ldots, X_n$ are independent as the Bernoulli trials are. Note that $\forall i = 1, \ldots, n$, $E\left[X_i\right] = 1 \times p + 0 \times q = p$, $E\left[X_i^2\right] = 1^2 \times p + 0^2 \times q = p$, so that $V\left[X_i\right] = p - p^2 = pq$.

Now the Binomial random variable X, denoting the total number of successes in these n Bernoulli trials, is nothing but $\sum_{i=1}^n X_i$ *i.e.* $X = \sum_{i=1}^n X_i$. By repeated use of ***Property E3*** of Expectation, given in Appendix B of the previous chapter, (with $c_1 = c_2 = 1$) we get

$$E[X] = E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E\left[X_i\right] = \sum_{i=1}^n p = np.$$

Since $X_1, \ldots, X_n$ are independent, $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$ and thus by repeated use of ***Property V2*** of Variance, given in Appendix B of the previous chapter, (with $a = b = 1$) we get

$$V[X] = V\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} V[X_i] = \sum_{i=1}^{n} pq = npq.$$

Having derived the mean and variance in three different ways, we wrap up our discussion about Binomial moments after computing its coefficients of skewness and kurtosis. For higher moment computation it is most convenient to derive them from the m.g.f.. Thus

$$M^{(3)}(t) = np\frac{d}{dt}\left\{e^t(q+pe^t)^{n-2}\left(q+npe^t\right)\right\} = npe^t(q+pe^t)^{n-3}\left\{q^2+(3n-1)pqe^t+n^2p^2e^{2t}\right\}$$

$$\Rightarrow E\left[X^3\right] = M^{(3)}(0) = np(q^2+(3n-1)pq+n^2p^2),$$

so that

$$\alpha_3 = E\left[(X-\mu)^3\right] = E\left[X^3\right] - 3\mu E\left[X^2\right] + 3\mu^2 E[X] - \mu^3 = npq(q-p)$$

and the coefficient of skewness $\beta_1 = (q-p)/\sqrt{npq}$. This coefficient of skewness makes a whole lot of intuitive sense. First notice that if $p = q = \frac{1}{2}$ the distribution is symmetric, and as expected $\alpha_3 = \beta_1 = 0$. If $q > p$, the values near 0 are more probable than values near n, making the distribution right-tailed and thus positively skewed. That is indeed the sign of α_3 and β_1 in this case. Similarly if $q < p$, the values near n are more probable than values near 0, making the distribution left-tailed and thus negatively skewed, and which indeed is the sign of α_3 and β_1 in this case.

$$\begin{aligned}
&M^{(4)}(t)\\
&= np\frac{d}{dt}\left\{e^t(q+pe^t)^{n-3}\left[q^2+(3n-1)pqe^t+n^2p^2e^{2t}\right]\right\}\\
&= npe^t(q+pe^t)^{n-4}\left\{q^3+(7n-4)pq^2e^t+\left(6n^2-4n+1\right)p^2qe^{2t}+n^3p^3e^{3t}\right\}\\
&\Rightarrow E\left[X^4\right] = np\left\{q^3+(7n-4)pq^2+\left(6n^2-4n+1\right)p^2q+n^3p^3\right\}
\end{aligned}$$

so that

$$\alpha_4 = E\left[(X-\mu)^4\right] = E\left[X^4\right] - 4\mu E\left[X^3\right] + 6\mu^2 E\left[X^2\right] - 4\mu^3 E[X] + \mu^4 = npq\left[q^2+(3n-4)pq+p^2\right]$$

and the coefficient of kurtosis $\beta_2 = \left[q^2+(3n-4)pq+p^2\right]/(npq)$. There is no hard and fast rule about when the distribution will be platokurtic($\beta_2 < 3$), mesokurtic($\beta_2 = 3$) or leptokurtic($\beta_2 > 3$). For instance, B(10,0.7) is platokurtic with $\beta_2 = 2.87619$, while B(10,0.788675) is mesokurtic with $\beta_2 = 3$ and B(10,0.8) is leptokurtic with $\beta_2 = 3.025$. However note that as $n \to \infty$, $\beta_2 \to 3$, indicating that for large n the peakedness of the Binomial distribution approaches that of the so-called "bell-curve" of the Normal distribution.

2.4 Quantiles

As mentioned in §2.1, the c.d.f. of the $B(n, p)$ distribution does not exist in closed-form. Thus unlike the moments, one does not have a closed-form expression for the π-th quantile of the $B(n, p)$ distribution for an arbitrary $\pi \in (0, 1)$. Instead one needs to use ***Definition 6*** of the previous chapter and Table I. Note that since $B(n, p)$ is a discrete distribution, care must be exercised for applying the definition of a quantile. We shall illustrate this quantile determination through a couple of examples.

Example 4.1 (Continued): Consider the multiple choice pop-quiz containing 20 questions with each question having 4 answer choices. Previously we had computed probability of an event involving a single student making random guesses among these four choices. Now consider the quiz given to a class with a large number of students, where everybody is unprepared and makes random guesses to the answers to the questions. Also suppose each correct answer gets 1 point and no penalty or negative marking for wrong answers. We now want to theoretically predict the features of the distribution of the points scored by the class in this quiz[2]. In particular we want answers to the following questions:

a. Depict the relative frequency of the scores.

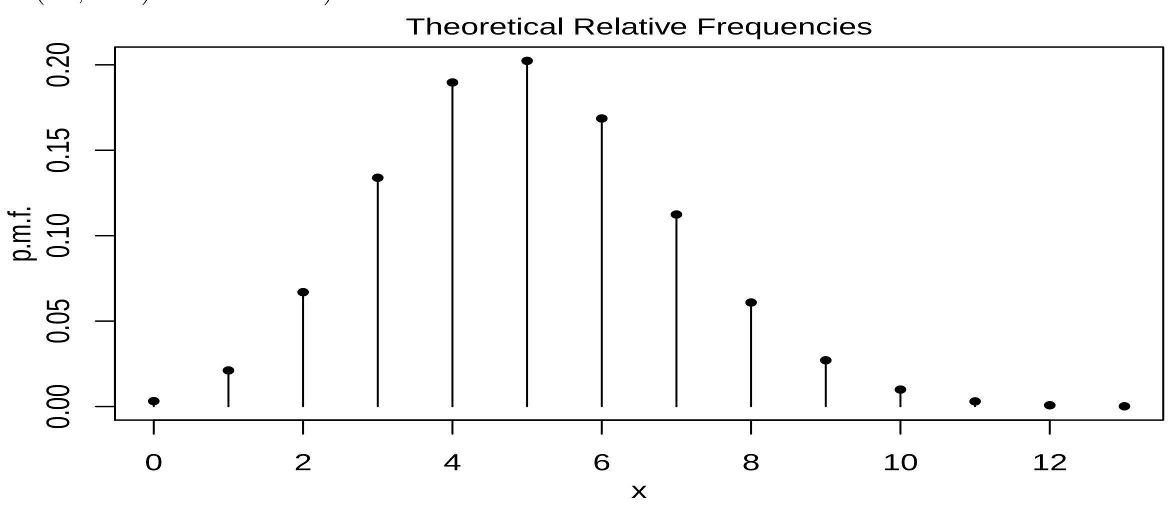
b. Find the mean and variance of the scores.

c. What is the score that a student is most likely to get?

d. Find the median and IQR of the scores.

e. If the cut-off for the letter grades A, B, C and D are 90-th, 80-th, 70-th and 60-th percentile of the scores respectively, determine the theoretical values of these cut-offs.

Solution (a): From earlier discussion we know the scores will follow a B(20,0.25) distribution and thus the (theoretical) relative frequencies of the marks scored would correspond to the respective p.m.f. values. Thus we calculate these p.m.f. values from Table I for $n = 20$, $p = 0.25$, and $x = 0, 1, \ldots, 13$ (as according to Table I, $P[X = x] \approx 0$ for $x = 14, \ldots, 20$ for the B(20,0.25) distribution) and then make a bar-chart out of these as follows:



[2]Note that given the points scored by all the students in the class, summarizing the distribution is a problem of Descriptive Statistics. Here in the absence of any data, using probability theory, we are trying to theoretically predict how such a data set might look like and the kind of summary measures it might have.

(b): We need not use the definitions involving lengthy arithmetic for computing the first and second raw and central moment. We know that in general if $X \sim \mathrm{B}(n,p)$, $E[X] = np$ and $V[X] = np(1-p)$ which for $n = 20$ and $p = 0.25$, yields $E[X] = 5$ and $V[X] = 3.75$. Thus the mean score of the class is predicted to be 5 with a variance of 3.75.
(c): The score that a student is most likely to get is given by the mode of the distribution. From the plot in **a** it is easy to spot that the mode is going to occur at 5. However recall that the plot was arrived at after (the lengthy calculation of) evaluating the p.m.f. at $x = 0, 1, \ldots, 13$. But there is no need to do this for answering this question. In §2.2 it was established that for a general $\mathrm{B}(n,p)$ distribution, if $(n+1)p$ is not an integer, its mode occurs at $\lfloor (n+1)p \rfloor$, the integer part of $(n+1)p$. For $n = 20$ and $p = 0.25$, this says that the mode occurs at $\lfloor 5.25 \rfloor = 5$. Thus the the score that a student is most likely to get is 5.
(d): Median is the 0.5-th quantile. According to ***Definition 6*** of the previous chapter, this means that we are after a number $\xi_{0.5}$, such that $F(\xi_{0.5}) \geq 0.5$ and $F(\xi_{0.5}-) \leq 0.5$, where $F(\cdot)$ is the c.d.f. of the B(20,0.25) distribution. The c.d.f. of the B(20,0.25) distribution is tabulated in Table 1, from which we see that $F(5) = 0.6172 \geq 0.5$ and $F(5-) = F(4) = 0.4148 \leq 0.5$. Thus the median is 5. IQR requires evaluation of $\xi_{0.25}$ and $\xi_{0.75}$, the 0.25-th and 0.75-th quantiles respectively. Since $F(4) = 0.4148 \geq 0.25$ and $F(4-) = F(3) = 0.2252 \leq 0.25$, $\xi_{0.25} = 4$; and similarly since $F(6) = 0.7858 \geq 0.75$ and $F(6-) = F(5) = 0.6172 \leq 0.75$, $\xi_{0.75} = 6$. Thus the IQR of the distribution is 2.
(d): This requires evaluation of $\xi_{0.9}$, $\xi_{0.8}$, $\xi_{0.7}$ and $\xi_{0.6}$, where ξ_π is the π-th quantile.

$$F(8) = 0.9591 \geq 0.9 \text{ and } F(8-) = F(7) = 0.8982 \leq 0.9 \Rightarrow \xi_{0.9} = 8$$

$$F(7) = 0.8982 \geq 0.8 \text{ and } F(7-) = F(6) = 0.7858 \leq 0.8 \Rightarrow \xi_{0.8} = 7$$

$$F(6) = 0.7858 \geq 0.7 \text{ and } F(6-) = F(5) = 0.6172 \leq 0.7 \Rightarrow \xi_{0.7} = 6$$

$$F(5) = 0.6172 \geq 0.6 \text{ and } F(5-) = F(4) = 0.4148 \leq 0.6 \Rightarrow \xi_{0.6} = 5$$

Thus the theoretical cut-offs for A, B, C and D are 8, 7, 6 and 5 respectively. ▽

3 Poisson Distribution

The Poisson distribution has already been introduced in **Example 3.27**, in terms of a discrete random variable having p.m.f. $p(x) = e^{-\lambda}\frac{\lambda^x}{x!}$ for $x = 0, 1, 2, \ldots$ for some parameter $\lambda > 0$. But this mathematical definition as such is not of much practical use, unless one understands how this formula of the p.m.f. comes about. A complicated derivation of this p.m.f. via its charactersitic function has already been provided in **Example 3.28** at the end of Chapter 3. Though the purpose of **Example 3.28** was to illustrate the concept of *convergence in distribution*, rather than the introduction of the Poisson r.v., the derivation there however contains the essence of the practical definition of the Poisson model, which may be described as follows.

Definition 4.3: Consider a sequence of independent Bernoulli trials in which p_n, the probability of Success, depends on n, the number of trials, in such a manner that np_n has a limit (say λ) as $n \to \infty$. Then as $n \to \infty$, X, the total number of Successes in such Bernoulli trials is said to have a **Poisson Distribution**, which is denoted by $X \sim \text{Poisson}(\lambda)$.

Before indulging into the unavoidable mathematical treatment for a transparent understanding of the Poisson distribution, let us first see how the above definition helps in modeling certain types of counts as Poisson r.v.. Consider for example the number of telephone calls an executive receives during a business day, or the number of typos in a page of a book, or the number of defects on the body of a car, or the number of customers arriving at a service counter during the first hour of its opening, or the number of TV sets sold by a particular electronic showroom on any given day, or the number of earth-quakes experienced every year at a point on a geological fault. All these random variables may be modeled as Poisson, since as required by ***Definition 4.3*** above, they may be conceptualised as number of Successes in a limiting sequence of Bernoulli trials with a large n and a small p with a moderate np. For instance for the number-of-customers-in-the-first-hour (say X) example, one can first imagine dividing the hour into a large number of small time intervals, so that in a given interval a customer either arrives (Success) or not (Failure) (with a diminishing probability of Success for smaller intervals), and then let X equal the limiting number of successes as the number of intervals goes to infinity. You must try imagining such limiting arguments for the other five examples above to get a feel for situations where Poisson models may be appropriate.

3.1 P.m.f. & C.d.f.

As mentioned in the begining of this chapter, all the discrete probability models in this chapter are introduced through their logical definitions rather than their mathemtical ones (which defines a distribution through its p.m.f. or c.d.f. or p.g.f. or m.g.f. or c.f. or such). While this approach is very useful for conceptual distribution recognition, quantitative questions pertaining to such discrete r.v. require one to have a handle on the mathemtical constructs of the distributions, which starts with the discussion of the p.m.f..

The connection of the p.m.f. of Poisson(λ), introduced in **Example 3.27**, to ***Definition 4.3*** may be understood through **Example 3.28**. According to ***Definition 4.3***, a Poisson(λ) r.v. is the "limit" of $B(n, p_n)$ r.v. with $\lim_{n\to\infty} np_n = \lambda$. The "limit" in question in this case pertains to the distributional limit. While distributional limits are typically checked via convergence of the characteristic functions (as in **Example 3.28**), in this case it may be directly verified as follows.

If $X \sim$B(n, p_n) with $\lim_{n\to\infty} np_n = \lambda$, then for a fixed non-negative integer x,

$$\begin{aligned}
\lim_{n\to\infty} P(X = x) &= \lim_{n\to\infty} \begin{pmatrix} n \\ x \end{pmatrix} p_n^x (1-p_n)^{n-x} \\
&= \lim_{n\to\infty} \left[\frac{1}{x!} \times \frac{n(n-1)\cdots(n-x+1)(n-x)!}{(n-x)!} \times \frac{(np_n)^x}{n^x} \times \left(1 - \frac{np_n}{n}\right)^{n-x} \right] \\
&= \frac{1}{x!} \times \left\{ \lim_{n\to\infty} 1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right\} \times \left(\lim_{n\to\infty} np_n\right)^x \times \lim_{n\to\infty}\left(1-\frac{np_n}{n}\right)^n \times \lim_{n\to\infty}(1-p_n)^{-x}
\end{aligned}$$

Now the term in $\{\cdot\}$ equals 1, $\lim_{n\to\infty}\left(1-\frac{np_n}{n}\right)^n = e^{-\lambda}$ since $\lim_{n\to\infty} np_n = \lambda$, and $\lim_{n\to\infty} p_n = 0$ as np_n has a limit. Therefore the above limit equals $e^{-\lambda}\frac{\lambda^x}{x!}$. Hence we obtain that for

$$X \sim \text{Poisson}(\lambda), \qquad P[X = x] = e^{-\lambda}\frac{\lambda^x}{x!} \text{ for } x = 0, 1, \ldots, \tag{2}$$

Like Binomial, the c.d.f. of Poisson, which is required for probability and quantile calculations, also does not have a readily computable formula except as a finite sum. Thus the c.d.f. of several Poisson(λ) distributions are tabulated in Table II of the Appendix. Following examples demonstrate the usage of this table as well as that of equation (2).

Example 4.4: It is estimated that on any given day 250,000 vehicles ply on a city's road. The probability of any one of these vehicles getting into an accident on that day is 10^{-5}. Assuming that the events of the vehicles getting into accidents are mutually independent of each other, find the probabilities that on any given day there are **a)** no, **b)** at most 5, and **c)** at least 2, vehicular accidents in that city.

Solution: If the random variable X denotes the number of vehicular accidents in that city on any given day, then because of the independence assumption, it is clear that $X \sim$ B$(250000, 10^{-5})$. Now as in ***Definition 4.3***, this situation may be viewed as one of a B(n, p_n), with a large n, small p_n and a "moderate" $\lambda = np_n = 2.5$, so that the distribution of X may be approximated by that of a Poisson(2.5) random variable. Thus using (2) the answer to **a** is $e^{-2.5} = 0.082085$. Note that one can also get it as 0.0821 using Table II. For **b** we need to find $P(X \leq 5)$ which is same as $F_X(5) = 0.9580$ from Table II, which also yields the answer to **c** given by $P(X \geq 2) = 1 - F_X(1) = 1 - 0.2873 = 0.7127$. $\triangledown$

Appendix: C.D.F. Tables

Table I: C.d.f. $F(x)$ of B(n, p) Distributions

n	x	p									
↓	↓	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
2	0	0.9025	0.8100	0.7225	0.6400	0.5625	0.4900	0.4225	0.3600	0.3025	0.2500
	1	0.9975	0.9900	0.9775	0.9600	0.9375	0.9100	0.8775	0.8400	0.7975	0.7500
3	0	0.8574	0.7290	0.6141	0.5120	0.4219	0.3430	0.2746	0.2160	0.1664	0.1250
	1	0.9928	0.9720	0.9392	0.8960	0.8438	0.7840	0.7182	0.6480	0.5748	0.5000
	2	0.9999	0.9990	0.9966	0.9920	0.9844	0.9730	0.9571	0.9360	0.9089	0.8750
4	0	0.8145	0.6561	0.5220	0.4096	0.3164	0.2401	0.1785	0.1296	0.0915	0.0625
	1	0.9860	0.9477	0.8905	0.8192	0.7383	0.6517	0.5630	0.4752	0.3910	0.3125
	2	0.9995	0.9963	0.9880	0.9728	0.9492	0.9163	0.8735	0.8208	0.7585	0.6875
	3	1.0000	0.9999	0.9995	0.9984	0.9961	0.9919	0.9850	0.9744	0.9590	0.9375
5	0	0.7738	0.5905	0.4437	0.3277	0.2373	0.1681	0.1160	0.0778	0.0503	0.0312
	1	0.9774	0.9185	0.8352	0.7373	0.6328	0.5282	0.4284	0.3370	0.2562	0.1875
	2	0.9988	0.9914	0.9734	0.9421	0.8965	0.8369	0.7648	0.6826	0.5931	0.5000
	3	1.0000	0.9995	0.9978	0.9933	0.9844	0.9692	0.9460	0.9130	0.8688	0.8125
	4	1.0000	1.0000	0.9999	0.9997	0.9990	0.9976	0.9947	0.9898	0.9815	0.9688
6	0	0.7351	0.5314	0.3771	0.2621	0.1780	0.1176	0.0754	0.0467	0.0277	0.0156
	1	0.9672	0.8857	0.7765	0.6554	0.5339	0.4202	0.3191	0.2333	0.1636	0.1094
	2	0.9978	0.9842	0.9527	0.9011	0.8306	0.7443	0.6471	0.5443	0.4415	0.3437
	3	0.9999	0.9987	0.9941	0.9830	0.9624	0.9295	0.8826	0.8208	0.7447	0.6562
	4	1.0000	0.9999	0.9996	0.9984	0.9954	0.9891	0.9777	0.9590	0.9308	0.8906
	5	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9982	0.9959	0.9917	0.9844
7	0	0.6983	0.4783	0.3206	0.2097	0.1335	0.0824	0.0490	0.0280	0.0152	0.0078
	1	0.9556	0.8503	0.7166	0.5767	0.4449	0.3294	0.2338	0.1586	0.1024	0.0625
	2	0.9962	0.9743	0.9262	0.8520	0.7564	0.6471	0.5323	0.4199	0.3164	0.2266
	3	0.9998	0.9973	0.9879	0.9667	0.9294	0.8740	0.8002	0.7102	0.6083	0.5000
	4	1.0000	0.9998	0.9988	0.9953	0.9871	0.9712	0.9444	0.9037	0.8471	0.7734
	5	1.0000	1.0000	0.9999	0.9996	0.9987	0.9962	0.9910	0.9812	0.9643	0.9375
	6	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9994	0.9984	0.9963	0.9922
8	0	0.6634	0.4305	0.2725	0.1678	0.1001	0.0576	0.0319	0.0168	0.0084	0.0039
	1	0.9428	0.8131	0.6572	0.5033	0.3671	0.2553	0.1691	0.1064	0.0632	0.0352
	2	0.9942	0.9619	0.8948	0.7969	0.6785	0.5518	0.4278	0.3154	0.2201	0.1445
	3	0.9996	0.9950	0.9786	0.9437	0.8862	0.8059	0.7064	0.5941	0.4770	0.3633
	4	1.0000	0.9996	0.9971	0.9896	0.9727	0.9420	0.8939	0.8263	0.7396	0.6367
	5	1.0000	1.0000	0.9998	0.9988	0.9958	0.9887	0.9747	0.9502	0.9115	0.8555
	6	1.0000	1.0000	1.0000	0.9999	0.9996	0.9987	0.9964	0.9915	0.9819	0.9648
	7	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998	0.9993	0.9983	0.9961
9	0	0.6302	0.3874	0.2316	0.1342	0.0751	0.0404	0.0207	0.0101	0.0046	0.0020
	1	0.9288	0.7748	0.5995	0.4362	0.3003	0.1960	0.1211	0.0705	0.0385	0.0195
	2	0.9916	0.9470	0.8591	0.7382	0.6007	0.4628	0.3373	0.2318	0.1495	0.0898
	3	0.9994	0.9917	0.9661	0.9144	0.8343	0.7297	0.6089	0.4826	0.3614	0.2539

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
9	4	1.0000	0.9991	0.9944	0.9804	0.9511	0.9012	0.8283	0.7334	0.6214	0.5000
	5	1.0000	0.9999	0.9994	0.9969	0.9900	0.9747	0.9464	0.9006	0.8342	0.7461
	6	1.0000	1.0000	1.0000	0.9997	0.9987	0.9957	0.9888	0.9750	0.9502	0.9102
	7	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9986	0.9962	0.9909	0.9805
	8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9980
10	0	0.5987	0.3487	0.1969	0.1074	0.0563	0.0282	0.0135	0.0060	0.0025	0.0010
	1	0.9139	0.7361	0.5443	0.3758	0.2440	0.1493	0.0860	0.0464	0.0233	0.0107
	2	0.9885	0.9298	0.8202	0.6778	0.5256	0.3828	0.2616	0.1673	0.0996	0.0547
	3	0.9990	0.9872	0.9500	0.8791	0.7759	0.6496	0.5138	0.3823	0.2660	0.1719
	4	0.9999	0.9984	0.9901	0.9672	0.9219	0.8497	0.7515	0.6331	0.5044	0.3770
	5	1.0000	0.9999	0.9986	0.9936	0.9803	0.9527	0.9051	0.8338	0.7384	0.6230
	6	1.0000	1.0000	0.9999	0.9991	0.9965	0.9894	0.9740	0.9452	0.8980	0.8281
	7	1.0000	1.0000	1.0000	0.9999	0.9996	0.9984	0.9952	0.9877	0.9726	0.9453
	8	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9983	0.9955	0.9893
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9990
11	0	0.5688	0.3138	0.1673	0.0859	0.0422	0.0198	0.0088	0.0036	0.0014	0.0005
	1	0.8981	0.6974	0.4922	0.3221	0.1971	0.1130	0.0606	0.0302	0.0139	0.0059
	2	0.9848	0.9104	0.7788	0.6174	0.4552	0.3127	0.2001	0.1189	0.0652	0.0327
	3	0.9984	0.9815	0.9306	0.8389	0.7133	0.5696	0.4256	0.2963	0.1911	0.1133
	4	0.9999	0.9972	0.9841	0.9496	0.8854	0.7897	0.6683	0.5328	0.3971	0.2744
	5	1.0000	0.9997	0.9973	0.9883	0.9657	0.9218	0.8513	0.7535	0.6331	0.5000
	6	1.0000	1.0000	0.9997	0.9980	0.9924	0.9784	0.9499	0.9006	0.8262	0.7256
	7	1.0000	1.0000	1.0000	0.9998	0.9988	0.9957	0.9878	0.9707	0.9390	0.8867
	8	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9980	0.9941	0.9852	0.9673
	9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9993	0.9978	0.9941
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9995
12	0	0.5404	0.2824	0.1422	0.0687	0.0317	0.0138	0.0057	0.0022	0.0008	0.0002
	1	0.8816	0.6590	0.4435	0.2749	0.1584	0.0850	0.0424	0.0196	0.0083	0.0032
	2	0.9804	0.8891	0.7358	0.5583	0.3907	0.2528	0.1513	0.0834	0.0421	0.0193
	3	0.9978	0.9744	0.9078	0.7946	0.6488	0.4925	0.3467	0.2253	0.1345	0.0730
	4	0.9998	0.9957	0.9761	0.9274	0.8424	0.7237	0.5833	0.4382	0.3044	0.1938
	5	1.0000	0.9995	0.9954	0.9806	0.9456	0.8822	0.7873	0.6652	0.5269	0.3872
	6	1.0000	0.9999	0.9993	0.9961	0.9857	0.9614	0.9154	0.8418	0.7393	0.6128
	7	1.0000	1.0000	0.9999	0.9994	0.9972	0.9905	0.9745	0.9427	0.8883	0.8062
	8	1.0000	1.0000	1.0000	0.9999	0.9996	0.9983	0.9944	0.9847	0.9644	0.9270
	9	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9992	0.9972	0.9921	0.9807
	10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9968
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9998
13	0	0.5133	0.2542	0.1209	0.0550	0.0238	0.0097	0.0037	0.0013	0.0004	0.0001
	1	0.8646	0.6213	0.3983	0.2336	0.1267	0.0637	0.0296	0.0126	0.0049	0.0017
	2	0.9755	0.8661	0.6920	0.5017	0.3326	0.2025	0.1132	0.0579	0.0269	0.0112
	3	0.9969	0.9658	0.8820	0.7473	0.5843	0.4206	0.2783	0.1686	0.0929	0.0461

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
13	4	0.9997	0.9935	0.9658	0.9009	0.7940	0.6543	0.5005	0.3530	0.2279	0.1334
	5	1.0000	0.9991	0.9925	0.9700	0.9198	0.8346	0.7159	0.5744	0.4268	0.2905
	6	1.0000	0.9999	0.9987	0.9930	0.9757	0.9376	0.8705	0.7712	0.6437	0.5000
	7	1.0000	1.0000	0.9998	0.9988	0.9944	0.9818	0.9538	0.9023	0.8212	0.7095
	8	1.0000	1.0000	1.0000	0.9998	0.9990	0.9960	0.9874	0.9679	0.9302	0.8666
	9	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9975	0.9922	0.9797	0.9539
	10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9987	0.9959	0.9888
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9983
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
14	0	0.4877	0.2288	0.1028	0.0440	0.0178	0.0068	0.0024	0.0008	0.0002	0.0001
	1	0.8470	0.5846	0.3567	0.1979	0.1010	0.0475	0.0205	0.0081	0.0029	0.0009
	2	0.9699	0.8416	0.6479	0.4481	0.2811	0.1608	0.0839	0.0398	0.0170	0.0065
	3	0.9958	0.9559	0.8535	0.6982	0.5213	0.3552	0.2205	0.1243	0.0632	0.0287
	4	0.9996	0.9908	0.9533	0.8702	0.7415	0.5842	0.4227	0.2793	0.1672	0.0898
	5	1.0000	0.9985	0.9885	0.9561	0.8883	0.7805	0.6405	0.4859	0.3373	0.2120
	6	1.0000	0.9998	0.9978	0.9884	0.9617	0.9067	0.8164	0.6925	0.5461	0.3953
	7	1.0000	1.0000	0.9997	0.9976	0.9897	0.9685	0.9247	0.8499	0.7414	0.6047
	8	1.0000	1.0000	1.0000	0.9996	0.9978	0.9917	0.9757	0.9417	0.8811	0.7880
	9	1.0000	1.0000	1.0000	1.0000	0.9997	0.9983	0.9940	0.9825	0.9574	0.9102
	10	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9989	0.9961	0.9886	0.9713
	11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9978	0.9935
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9991
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
15	0	0.4633	0.2059	0.0874	0.0352	0.0134	0.0047	0.0016	0.0005	0.0001	0.0000
	1	0.8290	0.5490	0.3186	0.1671	0.0802	0.0353	0.0142	0.0052	0.0017	0.0005
	2	0.9638	0.8159	0.6042	0.3980	0.2361	0.1268	0.0617	0.0271	0.0107	0.0037
	3	0.9945	0.9444	0.8227	0.6482	0.4613	0.2969	0.1727	0.0905	0.0424	0.0176
	4	0.9994	0.9873	0.9383	0.8358	0.6865	0.5155	0.3519	0.2173	0.1204	0.0592
	5	0.9999	0.9978	0.9832	0.9389	0.8516	0.7216	0.5643	0.4032	0.2608	0.1509
	6	1.0000	0.9997	0.9964	0.9819	0.9434	0.8689	0.7548	0.6098	0.4522	0.3036
	7	1.0000	1.0000	0.9994	0.9958	0.9827	0.9500	0.8868	0.7869	0.6535	0.5000
	8	1.0000	1.0000	0.9999	0.9992	0.9958	0.9848	0.9578	0.9050	0.8182	0.6964
	9	1.0000	1.0000	1.0000	0.9999	0.9992	0.9963	0.9876	0.9662	0.9231	0.8491
	10	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9972	0.9907	0.9745	0.9408
	11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9981	0.9937	0.9824
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9989	0.9963
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
16	0	0.4401	0.1853	0.0743	0.0281	0.0100	0.0033	0.0010	0.0003	0.0001	0.0000
	1	0.8108	0.5147	0.2839	0.1407	0.0635	0.0261	0.0098	0.0033	0.0010	0.0003
	2	0.9571	0.7892	0.5614	0.3518	0.1971	0.0994	0.0451	0.0183	0.0066	0.0021
	3	0.9930	0.9316	0.7899	0.5981	0.4050	0.2459	0.1339	0.0651	0.0281	0.0106

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
16	4	0.9991	0.9830	0.9209	0.7982	0.6302	0.4499	0.2892	0.1666	0.0853	0.0384
	5	0.9999	0.9967	0.9765	0.9183	0.8103	0.6598	0.4900	0.3288	0.1976	0.1051
	6	1.0000	0.9995	0.9944	0.9733	0.9204	0.8247	0.6881	0.5272	0.3660	0.2272
	7	1.0000	0.9999	0.9989	0.9930	0.9729	0.9256	0.8406	0.7161	0.5629	0.4018
	8	1.0000	1.0000	0.9998	0.9985	0.9925	0.9743	0.9329	0.8577	0.7441	0.5982
	9	1.0000	1.0000	1.0000	0.9998	0.9984	0.9929	0.9771	0.9417	0.8759	0.7728
	10	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9938	0.9809	0.9514	0.8949
	11	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987	0.9951	0.9851	0.9616
	12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9991	0.9965	0.9894
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9979
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
17	0	0.4181	0.1668	0.0631	0.0225	0.0075	0.0023	0.0007	0.0002	0.0000	0.0000
	1	0.7922	0.4818	0.2525	0.1182	0.0501	0.0193	0.0067	0.0021	0.0006	0.0001
	2	0.9497	0.7618	0.5198	0.3096	0.1637	0.0774	0.0327	0.0123	0.0041	0.0012
	3	0.9912	0.9174	0.7556	0.5489	0.3530	0.2019	0.1028	0.0464	0.0184	0.0064
	4	0.9988	0.9779	0.9013	0.7582	0.5739	0.3887	0.2348	0.1260	0.0596	0.0245
	5	0.9999	0.9953	0.9681	0.8943	0.7653	0.5968	0.4197	0.2639	0.1471	0.0717
	6	1.0000	0.9992	0.9917	0.9623	0.8929	0.7752	0.6188	0.4478	0.2902	0.1662
	7	1.0000	0.9999	0.9983	0.9891	0.9598	0.8954	0.7872	0.6405	0.4743	0.3145
	8	1.0000	1.0000	0.9997	0.9974	0.9876	0.9597	0.9006	0.8011	0.6626	0.5000
	9	1.0000	1.0000	1.0000	0.9995	0.9969	0.9873	0.9617	0.9081	0.8166	0.6855
	10	1.0000	1.0000	1.0000	0.9999	0.9994	0.9968	0.9880	0.9652	0.9174	0.8338
	11	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9970	0.9894	0.9699	0.9283
	12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9975	0.9914	0.9755
	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9981	0.9936
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9988
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
18	0	0.3972	0.1501	0.0536	0.0180	0.0056	0.0016	0.0004	0.0001	0.0000	0.0000
	1	0.7735	0.4503	0.2241	0.0991	0.0395	0.0142	0.0046	0.0013	0.0003	0.0001
	2	0.9419	0.7338	0.4797	0.2713	0.1353	0.0600	0.0236	0.0082	0.0025	0.0007
	3	0.9891	0.9018	0.7202	0.5010	0.3057	0.1646	0.0783	0.0328	0.0120	0.0038
	4	0.9985	0.9718	0.8794	0.7164	0.5187	0.3327	0.1886	0.0942	0.0411	0.0154
	5	0.9998	0.9936	0.9581	0.8671	0.7175	0.5344	0.3550	0.2088	0.1077	0.0481
	6	1.0000	0.9988	0.9882	0.9487	0.8610	0.7217	0.5491	0.3743	0.2258	0.1189
	7	1.0000	0.9998	0.9973	0.9837	0.9431	0.8593	0.7283	0.5634	0.3915	0.2403
	8	1.0000	1.0000	0.9995	0.9957	0.9807	0.9404	0.8609	0.7368	0.5778	0.4073
	9	1.0000	1.0000	0.9999	0.9991	0.9946	0.9790	0.9403	0.8653	0.7473	0.5927
	10	1.0000	1.0000	1.0000	0.9998	0.9988	0.9939	0.9788	0.9424	0.8720	0.7597
	11	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9938	0.9797	0.9463	0.8811
	12	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9986	0.9942	0.9817	0.9519

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
18	13	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987	0.9951	0.9846
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9990	0.9962
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
19	0	0.3774	0.1351	0.0456	0.0144	0.0042	0.0011	0.0003	0.0001	0.0000	0.0000
	1	0.7547	0.4203	0.1985	0.0829	0.0310	0.0104	0.0031	0.0008	0.0002	0.0000
	2	0.9335	0.7054	0.4413	0.2369	0.1113	0.0462	0.0170	0.0055	0.0015	0.0004
	3	0.9868	0.8850	0.6841	0.4551	0.2631	0.1332	0.0591	0.0230	0.0077	0.0022
	4	0.9980	0.9648	0.8556	0.6733	0.4654	0.2822	0.1500	0.0696	0.0280	0.0096
	5	0.9998	0.9914	0.9463	0.8369	0.6678	0.4739	0.2968	0.1629	0.0777	0.0318
	6	1.0000	0.9983	0.9837	0.9324	0.8251	0.6655	0.4812	0.3081	0.1727	0.0835
	7	1.0000	0.9997	0.9959	0.9767	0.9225	0.8180	0.6656	0.4878	0.3169	0.1796
	8	1.0000	1.0000	0.9992	0.9933	0.9713	0.9161	0.8145	0.6675	0.4940	0.3238
	9	1.0000	1.0000	0.9999	0.9984	0.9911	0.9674	0.9125	0.8139	0.6710	0.5000
	10	1.0000	1.0000	1.0000	0.9997	0.9977	0.9895	0.9653	0.9115	0.8159	0.6762
	11	1.0000	1.0000	1.0000	1.0000	0.9995	0.9972	0.9886	0.9648	0.9129	0.8204
	12	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9969	0.9884	0.9658	0.9165
	13	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9969	0.9891	0.9682
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9972	0.9904
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9978
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	0	0.3585	0.1216	0.0388	0.0115	0.0032	0.0008	0.0002	0.0000	0.0000	0.0000
	1	0.7358	0.3917	0.1756	0.0692	0.0243	0.0076	0.0021	0.0005	0.0001	0.0000
	2	0.9245	0.6769	0.4049	0.2061	0.0913	0.0355	0.0121	0.0036	0.0009	0.0002
	3	0.9841	0.8670	0.6477	0.4114	0.2252	0.1071	0.0444	0.0160	0.0049	0.0013
	4	0.9974	0.9568	0.8298	0.6296	0.4148	0.2375	0.1182	0.0510	0.0189	0.0059
	5	0.9997	0.9887	0.9327	0.8042	0.6172	0.4164	0.2454	0.1256	0.0553	0.0207
	6	1.0000	0.9976	0.9781	0.9133	0.7858	0.6080	0.4166	0.2500	0.1299	0.0577
	7	1.0000	0.9996	0.9941	0.9679	0.8982	0.7723	0.6010	0.4159	0.2520	0.1316
	8	1.0000	0.9999	0.9987	0.9900	0.9591	0.8867	0.7624	0.5956	0.4143	0.2517
	9	1.0000	1.0000	0.9998	0.9974	0.9861	0.9520	0.8782	0.7553	0.5914	0.4119
	10	1.0000	1.0000	1.0000	0.9994	0.9961	0.9829	0.9468	0.8725	0.7507	0.5881
	11	1.0000	1.0000	1.0000	0.9999	0.9991	0.9949	0.9804	0.9435	0.8692	0.7483
	12	1.0000	1.0000	1.0000	1.0000	0.9998	0.9987	0.9940	0.9790	0.9420	0.8684
	13	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9935	0.9786	0.9423
	14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9936	0.9793
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9985	0.9941
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9987
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
21	0	0.3406	0.1094	0.0329	0.0092	0.0024	0.0006	0.0001	0.0000	0.0000	0.0000
	1	0.7170	0.3647	0.1550	0.0576	0.0190	0.0056	0.0014	0.0003	0.0001	0.0000
	2	0.9151	0.6484	0.3705	0.1787	0.0745	0.0271	0.0086	0.0024	0.0006	0.0001
	3	0.9811	0.8480	0.6113	0.3704	0.1917	0.0856	0.0331	0.0110	.0031	0.0007
	4	0.9968	0.9478	0.8025	0.5860	0.3674	0.1984	0.0924	0.0370	0.0126	0.0036
	5	0.9996	0.9856	0.9173	0.7693	0.5666	0.3627	0.2009	0.0957	0.0389	0.0133
	6	1.0000	0.9967	0.9713	0.8915	0.7436	0.5505	0.3567	0.2002	0.0964	0.0392
	7	1.0000	0.9994	0.9917	0.9569	0.8701	0.7230	0.5365	0.3495	0.1971	0.0946
	8	1.0000	0.9999	0.9980	0.9856	0.9439	0.8523	0.7059	0.5237	0.3413	0.1917
	9	1.0000	1.0000	0.9996	0.9959	0.9794	0.9324	0.8377	0.6914	0.5117	0.3318
	10	1.0000	1.0000	0.9999	0.9990	0.9936	0.9736	0.9228	0.8256	0.6790	0.5000
	11	1.0000	1.0000	1.0000	0.9998	0.9983	0.9913	0.9687	0.9151	0.8159	0.6682
	12	1.0000	1.0000	1.0000	1.0000	0.9996	0.9976	0.9892	0.9648	0.9092	0.8083
	13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9969	0.9877	0.9621	0.9054
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9964	0.9868	0.9608
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9963	0.9867
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9992	0.9964
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
22	0	0.3235	0.0985	0.0280	0.0074	0.0018	0.0004	0.0001	0.0000	0.0000	0.0000
	1	0.6982	0.3392	0.1367	0.0480	0.0149	0.0041	0.0010	0.0002	0.0000	0.0000
	2	0.9052	0.6200	0.3382	0.1545	0.0606	0.0207	0.0061	0.0016	0.0003	0.0001
	3	0.9778	0.8281	0.5752	0.3320	0.1624	0.0681	0.0245	0.0076	0.0020	0.0004
	4	0.9960	0.9379	0.7738	0.5429	0.3235	0.1645	0.0716	0.0266	0.0083	0.0022
	5	0.9994	0.9818	0.9001	0.7326	0.5168	0.3134	0.1629	0.0722	0.0271	0.0085
	6	0.9999	0.9956	0.9632	0.8670	0.6994	0.4942	0.3022	0.1584	0.0705	0.0262
	7	1.0000	0.9991	0.9886	0.9439	0.8385	0.6713	0.4736	0.2898	0.1518	0.0669
	8	1.0000	0.9999	0.9970	0.9799	0.9254	0.8135	0.6466	0.4540	0.2764	0.1431
	9	1.0000	1.0000	0.9993	0.9939	0.9705	0.9084	0.7916	0.6244	0.4350	0.2617
	10	1.0000	1.0000	0.9999	0.9984	0.9900	0.9613	0.8930	0.7720	0.6037	0.4159
	11	1.0000	1.0000	1.0000	0.9997	0.9971	0.9860	0.9526	0.8793	0.7543	0.5841
	12	1.0000	1.0000	1.0000	0.9999	0.9993	0.9957	0.9820	0.9449	0.8672	0.7383
	13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9989	0.9942	0.9785	0.9383	0.8569
	14	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9984	0.9930	0.9757	0.9331
	15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9981	0.9920	0.9738
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9979	0.9915
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9978
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
23	0	0.3074	0.0886	0.0238	0.0059	0.0013	0.0003	0.0000	0.0000	0.0000	0.0000
	1	0.6794	0.3151	0.1204	0.0398	0.0116	0.0030	0.0007	0.0001	0.0000	0.0000
	2	0.8948	0.5920	0.3080	0.1332	0.0492	0.0157	0.0043	0.0010	0.0002	0.0000
	3	0.9742	0.8073	0.5396	0.2965	0.1370	0.0538	0.0181	0.0052	0.0012	0.0002
	4	0.9951	0.9269	0.7440	0.5007	0.2832	0.1356	0.0551	0.0190	0.0055	0.0013
	5	0.9992	0.9774	0.8811	0.6947	0.4685	0.2688	0.1309	0.0540	0.0186	0.0053
	6	0.9999	0.9942	0.9537	0.8402	0.6537	0.4399	0.2534	0.1240	0.0510	0.0173
	7	1.0000	0.9988	0.9848	0.9285	0.8037	0.6181	0.4136	0.2373	0.1152	0.0466
	8	1.0000	0.9998	0.9958	0.9727	0.9037	0.7709	0.5860	0.3884	0.2203	0.1050
	9	1.0000	1.0000	0.9990	0.9911	0.9592	0.8799	0.7408	0.5562	0.3636	0.2024
	10	1.0000	1.0000	0.9998	0.9975	0.9851	0.9454	0.8575	0.7129	0.5278	0.3388
	11	1.0000	1.0000	1.0000	0.9994	0.9954	0.9786	0.9318	0.8364	0.6865	0.5000
	12	1.0000	1.0000	1.0000	0.9999	0.9988	0.9928	0.9717	0.9187	0.8164	0.6612
	13	1.0000	1.0000	1.0000	1.0000	0.9997	0.9979	0.9900	0.9651	0.9063	0.7976
	14	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9970	0.9872	0.9589	0.8950
	15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9960	0.9847	0.9534
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9990	0.9952	0.9827
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988	0.9947
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9987
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
24	0	0.2920	0.0798	0.0202	0.0047	0.0010	0.0002	0.0000	0.0000	0.0000	0.0000
	1	0.6608	0.2925	0.1059	0.0331	0.0090	0.0022	0.0005	0.0001	0.0000	0.0000
	2	0.8841	0.5643	0.2798	0.1145	0.0398	0.0119	0.0030	0.0007	0.0001	0.0000
	3	0.9702	0.7857	0.5049	0.2639	0.1150	0.0424	0.0133	0.0035	0.0008	0.0001
	4	0.9940	0.9149	0.7134	0.4599	0.2466	0.1111	0.0422	0.0134	0.0036	0.0008
	5	0.9990	0.9723	0.8606	0.6559	0.4222	0.2288	0.1044	0.0400	0.0127	0.0033
	6	0.9999	0.9925	0.9428	0.8111	0.6074	0.3886	0.2106	0.0960	0.0364	0.0113
	7	1.0000	0.9983	0.9801	0.9108	0.7662	0.5647	0.3575	0.1919	0.0863	0.0320
	8	1.0000	0.9997	0.9941	0.9638	0.8787	0.7250	0.5257	0.3279	0.1730	0.0758
	9	1.0000	0.9999	0.9985	0.9874	0.9453	0.8472	0.6866	0.4891	0.2991	0.1537
	10	1.0000	1.0000	0.9997	0.9962	0.9787	0.9258	0.8167	0.6502	0.4539	0.2706
	11	1.0000	1.0000	0.9999	0.9990	0.9928	0.9686	0.9058	0.7870	0.6151	0.4194
	12	1.0000	1.0000	1.0000	0.9998	0.9979	0.9885	0.9577	0.8857	0.7580	0.5806
	13	1.0000	1.0000	1.0000	1.0000	0.9995	0.9964	0.9836	0.9465	0.8659	0.7294
	14	1.0000	1.0000	1.0000	1.0000	0.9999	0.9990	0.9945	0.9783	0.9352	0.8463
	15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9984	0.9925	0.9731	0.9242
	16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9978	0.9905	0.9680
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9972	0.9887
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9967
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
25	0	0.2774	0.0718	0.0172	0.0038	0.0008	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.6424	0.2712	0.0931	0.0274	0.0070	0.0016	0.0003	0.0001	0.0000	0.0000
	2	0.8729	0.5371	0.2537	0.0982	0.0321	0.0090	0.0021	0.0004	0.0001	0.0000
	3	0.9659	0.7636	0.4711	0.2340	0.0962	0.0332	0.0097	0.0024	0.0005	0.0001
	4	0.9928	0.9020	0.6821	0.4207	0.2137	0.0905	0.0320	0.0095	0.0023	0.0005
	5	0.9988	0.9666	0.8385	0.6167	0.3783	0.1935	0.0826	0.0294	0.0086	0.0020
	6	0.9998	0.9905	0.9305	0.7800	0.5611	0.3407	0.1734	0.0736	0.0258	0.0073
	7	1.0000	0.9977	0.9745	0.8909	0.7265	0.5118	0.3061	0.1536	0.0639	0.0216
	8	1.0000	0.9995	0.9920	0.9532	0.8506	0.6769	0.4668	0.2735	0.1340	0.0539
	9	1.0000	0.9999	0.9979	0.9827	0.9287	0.8106	0.6303	0.4246	0.2424	0.1148
	10	1.0000	1.0000	0.9995	0.9944	0.9703	0.9022	0.7712	0.5858	0.3843	0.2122
	11	1.0000	1.0000	0.9999	0.9985	0.9893	0.9558	0.8746	0.7323	0.5426	0.3450
	12	1.0000	1.0000	1.0000	0.9996	0.9966	0.9825	0.9396	0.8462	0.6937	0.5000
	13	1.0000	1.0000	1.0000	0.9999	0.9991	0.9940	0.9745	0.9222	0.8173	0.6550
	14	1.0000	1.0000	1.0000	1.0000	0.9998	0.9982	0.9907	0.9656	0.9040	0.7878
	15	1.0000	1.0000	1.0000	1.0000	1.0000	0.9995	0.9971	0.9868	0.9560	0.8852
	16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9957	0.9826	0.9461
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988	0.9942	0.9784
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9984	0.9927
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9996	0.9980
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
26	0	0.2635	0.0646	0.0146	0.0030	0.0006	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.6241	0.2513	0.0817	0.0227	0.0055	0.0011	0.0002	0.0000	0.0000	0.0000
	2	0.8614	0.5105	0.2296	0.0841	0.0258	0.0067	0.0015	0.0003	0.0000	0.0000
	3	0.9613	0.7409	0.4385	0.2068	0.0802	0.0260	0.0070	0.0016	0.0003	0.0000
	4	0.9915	0.8882	0.6505	0.3833	0.1844	0.0733	0.0242	0.0066	0.0015	0.0003
	5	0.9985	0.9601	0.8150	0.5775	0.3371	0.1626	0.0649	0.0214	0.0058	0.0012
	6	0.9998	0.9881	0.9167	0.7474	0.5154	0.2965	0.1416	0.0559	0.0180	0.0047
	7	1.0000	0.9970	0.9679	0.8687	0.6852	0.4605	0.2596	0.1216	0.0467	0.0145
	8	1.0000	0.9994	0.9894	0.9408	0.8195	0.6274	0.4106	0.2255	0.1024	0.0378
	9	1.0000	0.9999	0.9970	0.9768	0.9091	0.7705	0.5731	0.3642	0.1936	0.0843
	10	1.0000	1.0000	0.9993	0.9921	0.9599	0.8747	0.7219	0.5213	0.3204	0.1635
	11	1.0000	1.0000	0.9998	0.9977	0.9845	0.9397	0.8384	0.6737	0.4713	0.2786
	12	1.0000	1.0000	1.0000	0.9994	0.9948	0.9745	0.9168	0.8007	0.6257	0.4225
	13	1.0000	1.0000	1.0000	0.9999	0.9985	0.9906	0.9623	0.8918	0.7617	0.5775
	14	1.0000	1.0000	1.0000	1.0000	0.9996	0.9970	0.9850	0.9482	0.8650	0.7214
	15	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991	0.9948	0.9783	0.9326	0.8365
	16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9985	0.9921	0.9707	0.9157
	17	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9975	0.9890	0.9622
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9965	0.9855

n	x	p									
↓	↓	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
26	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991	0.9953
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
27	0	0.2503	0.0581	0.0124	0.0024	0.0004	0.0001	0.0000	0.0000	0.0000	0.0000
	1	0.6061	0.2326	0.0716	0.0187	0.0042	0.0008	0.0001	0.0000	0.0000	0.0000
	2	0.8495	0.4846	0.2074	0.0718	0.0207	0.0051	0.0010	0.0002	0.0000	0.0000
	3	0.9563	0.7179	0.4072	0.1823	0.0666	0.0202	0.0051	0.0011	0.0002	0.0000
	4	0.9900	0.8734	0.6187	0.3480	0.1583	0.0591	0.0182	0.0046	0.0009	0.0002
	5	0.9981	0.9529	0.7903	0.5387	0.2989	0.1358	0.0507	0.0155	0.0038	0.0008
	6	0.9997	0.9853	0.9014	0.7134	0.4708	0.2563	0.1148	0.0421	0.0125	0.0030
	7	1.0000	0.9961	0.9602	0.8444	0.6427	0.4113	0.2183	0.0953	0.0338	0.0096
	8	1.0000	0.9991	0.9862	0.9263	0.7859	0.5773	0.3577	0.1839	0.0774	0.0261
	9	1.0000	0.9998	0.9958	0.9696	0.8867	0.7276	0.5162	0.3087	0.1526	0.0610
	10	1.0000	1.0000	0.9989	0.9890	0.9472	0.8434	0.6698	0.4585	0.2633	0.1239
	11	1.0000	1.0000	0.9998	0.9965	0.9784	0.9202	0.7976	0.6127	0.4034	0.2210
	12	1.0000	1.0000	1.0000	0.9990	0.9922	0.9641	0.8894	0.7499	0.5562	0.3506
	13	1.0000	1.0000	1.0000	0.9998	0.9976	0.9857	0.9464	0.8553	0.7005	0.5000
	14	1.0000	1.0000	1.0000	1.0000	0.9993	0.9950	0.9771	0.9257	0.8185	0.6494
	15	1.0000	1.0000	1.0000	1.0000	0.9998	0.9985	0.9914	0.9663	0.9022	0.7790
	16	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9972	0.9866	0.9536	0.8761
	17	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9954	0.9807	0.9390
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9931	0.9739
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9979	0.9904
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995	0.9970
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
28	0	0.2378	0.0523	0.0106	0.0019	0.0003	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5883	0.2152	0.0627	0.0155	0.0033	0.0006	0.0001	0.0000	0.0000	0.0000
	2	0.8373	0.4594	0.1871	0.0612	0.0166	0.0038	0.0007	0.0001	0.0000	0.0000
	3	0.9509	0.6946	0.3772	0.1602	0.0551	0.0157	0.0037	0.0007	0.0001	0.0000
	4	0.9883	0.8579	0.5869	0.3149	0.1354	0.0474	0.0136	0.0032	0.0006	0.0001
	5	0.9977	0.9450	0.7646	0.5005	0.2638	0.1128	0.0393	0.0111	0.0025	0.0005
	6	0.9996	0.9821	0.8848	0.6784	0.4279	0.2202	0.0923	0.0315	0.0086	0.0019
	7	1.0000	0.9950	0.9514	0.8182	0.5997	0.3648	0.1821	0.0740	0.0242	0.0063
	8	1.0000	0.9988	0.9823	0.9100	0.7501	0.5275	0.3089	0.1485	0.0578	0.0178
	9	1.0000	0.9998	0.9944	0.9609	0.8615	0.6825	0.4607	0.2588	0.1187	0.0436
	10	1.0000	1.0000	0.9985	0.9851	0.9321	0.8087	0.6160	0.3986	0.2135	0.0925
	11	1.0000	1.0000	0.9996	0.9950	0.9706	0.8972	0.7529	0.5510	0.3404	0.1725
	12	1.0000	1.0000	0.9999	0.9985	0.9888	0.9509	0.8572	0.6950	0.4875	0.2858
	13	1.0000	1.0000	1.0000	0.9996	0.9962	0.9792	0.9264	0.8132	0.6356	0.4253

n ↓	x ↓	p 0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
28	14	1.0000	1.0000	1.0000	0.9999	0.9989	0.9923	0.9663	0.8975	0.7654	0.5747
	15	1.0000	1.0000	1.0000	1.0000	0.9997	0.9975	0.9864	0.9501	0.8645	0.7142
	16	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9952	0.9785	0.9304	0.8275
	17	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9985	0.9919	0.9685	0.9075
	18	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9973	0.9875	0.9564
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9957	0.9822
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988	0.9937
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9981
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9995
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
29	0	0.2259	0.0471	0.0090	0.0015	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5708	0.1989	0.0549	0.0128	0.0025	0.0004	0.0001	0.0000	0.0000	0.0000
	2	0.8249	0.4350	0.1684	0.0520	0.0133	0.0028	0.0005	0.0001	0.0000	0.0000
	3	0.9452	0.6710	0.3487	0.1404	0.0455	0.0121	0.0026	0.0005	0.0001	0.0000
	4	0.9864	0.8416	0.5555	0.2839	0.1153	0.0379	0.0101	0.0022	0.0004	0.0001
	5	0.9973	0.9363	0.7379	0.4634	0.2317	0.0932	0.0303	0.0080	0.0017	0.0003
	6	0.9995	0.9784	0.8667	0.6429	0.3868	0.1880	0.0738	0.0233	0.0059	0.0012
	7	0.9999	0.9938	0.9414	0.7903	0.5568	0.3214	0.1507	0.0570	0.0172	0.0041
	8	1.0000	0.9984	0.9777	0.8916	0.7125	0.4787	0.2645	0.1187	0.0427	0.0121
	9	1.0000	0.9997	0.9926	0.9507	0.8337	0.6360	0.4076	0.2147	0.0913	0.0307
	10	1.0000	0.9999	0.9978	0.9803	0.9145	0.7708	0.5617	0.3427	0.1708	0.0680
	11	1.0000	1.0000	0.9995	0.9931	0.9610	0.8706	0.7050	0.4900	0.2833	0.1325
	12	1.0000	1.0000	0.9999	0.9978	0.9842	0.9348	0.8207	0.6374	0.4213	0.2291
	13	1.0000	1.0000	1.0000	0.9994	0.9944	0.9707	0.9022	0.7659	0.5689	0.3555
	14	1.0000	1.0000	1.0000	0.9999	0.9982	0.9883	0.9524	0.8638	0.7070	0.5000
	15	1.0000	1.0000	1.0000	1.0000	0.9995	0.9959	0.9794	0.9290	0.8199	0.6445
	16	1.0000	1.0000	1.0000	1.0000	0.9999	0.9987	0.9921	0.9671	0.9008	0.7709
	17	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997	0.9973	0.9865	0.9514	0.8675
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9992	0.9951	0.9790	0.9320
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9985	0.9920	0.9693
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9974	0.9879
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993	0.9959
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9988
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9997
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999
	25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n	x	p									
↓	↓	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
30	0	0.2146	0.0424	0.0076	0.0012	0.0002	0.0000	0.0000	0.0000	0.0000	0.0000
	1	0.5535	0.1837	0.0480	0.0105	0.0020	0.0003	0.0000	0.0000	0.0000	0.0000
	2	0.8122	0.4114	0.1514	0.0442	0.0106	0.0021	0.0003	0.0000	0.0000	0.0000
	3	0.9392	0.6474	0.3217	0.1227	0.0374	0.0093	0.0019	0.0003	0.0000	0.0000
	4	0.9844	0.8245	0.5245	0.2552	0.0979	0.0302	0.0075	0.0015	0.0002	0.0000
	5	0.9967	0.9268	0.7106	0.4275	0.2026	0.0766	0.0233	0.0057	0.0011	0.0002
	6	0.9994	0.9742	0.8474	0.6070	0.3481	0.1595	0.0586	0.0172	0.0040	0.0007
	7	0.9999	0.9922	0.9302	0.7608	0.5143	0.2814	0.1238	0.0435	0.0121	0.0026
	8	1.0000	0.9980	0.9722	0.8713	0.6736	0.4315	0.2247	0.0940	0.0312	0.0081
	9	1.0000	0.9995	0.9903	0.9389	0.8034	0.5888	0.3575	0.1763	0.0694	0.0214
	10	1.0000	0.9999	0.9971	0.9744	0.8943	0.7304	0.5078	0.2915	0.1350	0.0494
	11	1.0000	1.0000	0.9992	0.9905	0.9493	0.8407	0.6548	0.4311	0.2327	0.1002
	12	1.0000	1.0000	0.9998	0.9969	0.9784	0.9155	0.7802	0.5785	0.3592	0.1808
	13	1.0000	1.0000	1.0000	0.9991	0.9918	0.9599	0.8737	0.7145	0.5025	0.2923
	14	1.0000	1.0000	1.0000	0.9998	0.9973	0.9831	0.9348	0.8246	0.6448	0.4278
	15	1.0000	1.0000	1.0000	0.9999	0.9992	0.9936	0.9699	0.9029	0.7691	0.5722
	16	1.0000	1.0000	1.0000	1.0000	0.9998	0.9979	0.9876	0.9519	0.8644	0.7077
	17	1.0000	1.0000	1.0000	1.0000	0.9999	0.9994	0.9955	0.9788	0.9286	0.8192
	18	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9986	0.9917	0.9666	0.8998
	19	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9971	0.9862	0.9506
	20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9991	0.9950	0.9786
	21	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9984	0.9919
	22	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9996	0.9974
	23	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993
	24	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998
	25	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table II: C.d.f. $F(x)$ of Poisson(λ) Distributions

$\lambda \rightarrow$ $x \downarrow$	0.0001	0.0005	0.001	0.005	0.01	0.02	0.03	0.04	0.05	0.06
0	0.9999	0.9995	0.9990	0.9950	0.9900	0.9802	0.9704	0.9608	0.9512	0.9418
1	1.0000	1.0000	1.0000	1.0000	1.0000	0.9998	0.9996	0.9992	0.9988	0.9983

$\lambda \rightarrow$ $x \downarrow$	0.07	0.08	0.09	0.1	0.2	0.3	0.4	0.5	0.6	0.7
0	0.9324	0.9231	0.9139	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966
1	0.9977	0.9970	0.9962	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442
2	0.9999	0.9999	0.9999	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659
3	1.0000	1.0000	1.0000	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942

$\lambda \rightarrow$ $x \downarrow$	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0	0.4493	0.4066	0.3679	0.3329	0.3012	0.2725	0.2466	0.2231	0.2019	0.1827
1	0.8088	0.7725	0.7358	0.6990	0.6626	0.6268	0.5918	0.5578	0.5249	0.4932
2	0.9526	0.9371	0.9197	0.9004	0.8795	0.8571	0.8335	0.8088	0.7834	0.7572
3	0.9909	0.9865	0.9810	0.9743	0.9662	0.9569	0.9463	0.9344	0.9212	0.9068
4	0.9986	0.9977	0.9963	0.9946	0.9923	0.9893	0.9857	0.9814	0.9763	0.9704
5	0.9998	0.9997	0.9994	0.9990	0.9985	0.9978	0.9968	0.9955	0.9940	0.9920
6	1.0000	1.0000	0.9999	0.9999	0.9997	0.9996	0.9994	0.9991	0.9987	0.9981

$\lambda \rightarrow$ $x \downarrow$	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7
0	0.1653	0.1496	0.1353	0.1225	0.1108	0.1003	0.0907	0.0821	0.0743	0.0672
1	0.4628	0.4337	0.4060	0.3796	0.3546	0.3309	0.3084	0.2873	0.2674	0.2487
2	0.7306	0.7037	0.6767	0.6496	0.6227	0.5960	0.5697	0.5438	0.5184	0.4936
3	0.8913	0.8747	0.8571	0.8386	0.8194	0.7993	0.7787	0.7576	0.7360	0.7141
4	0.9636	0.9559	0.9473	0.9379	0.9275	0.9162	0.9041	0.8912	0.8774	0.8629
5	0.9896	0.9868	0.9834	0.9796	0.9751	0.9700	0.9643	0.9580	0.9510	0.9433
6	0.9974	0.9966	0.9955	0.9941	0.9925	0.9906	0.9884	0.9858	0.9828	0.9794
7	0.9994	0.9992	0.9989	0.9985	0.9980	0.9974	0.9967	0.9958	0.9947	0.9934
8	0.9999	0.9998	0.9998	0.9997	0.9995	0.9994	0.9991	0.9989	0.9985	0.9981
9	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9998	0.9997	0.9996	0.9995
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999

$\lambda \rightarrow$ $x \downarrow$	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7
0	0.0608	0.0550	0.0498	0.0450	0.0408	0.0369	0.0334	0.0302	0.0273	0.0247
1	0.2311	0.2146	0.1991	0.1847	0.1712	0.1586	0.1468	0.1359	0.1257	0.1162
2	0.4695	0.4460	0.4232	0.4012	0.3799	0.3594	0.3397	0.3208	0.3027	0.2854
3	0.6919	0.6696	0.6472	0.6248	0.6025	0.5803	0.5584	0.5366	0.5152	0.4942
4	0.8477	0.8318	0.8153	0.7982	0.7806	0.7626	0.7442	0.7254	0.7064	0.6872
5	0.9349	0.9258	0.9161	0.9057	0.8946	0.8829	0.8705	0.8576	0.8441	0.8301
6	0.9756	0.9713	0.9665	0.9612	0.9554	0.9490	0.9421	0.9347	0.9267	0.9182
7	0.9919	0.9901	0.9881	0.9858	0.9832	0.9802	0.9769	0.9733	0.9692	0.9648
8	0.9976	0.9969	0.9962	0.9953	0.9943	0.9931	0.9917	0.9901	0.9883	0.9863

$\lambda \rightarrow$ / $x \downarrow$	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5	3.6	3.7
9	0.9993	0.9991	0.9989	0.9986	0.9982	0.9978	0.9973	0.9967	0.9960	0.9952
10	0.9998	0.9998	0.9997	0.9996	0.9995	0.9994	0.9992	0.9990	0.9987	0.9984
11	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9996	0.9995
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999

$\lambda \rightarrow$ / $x \downarrow$	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5	4.6	4.7
0	0.0224	0.0202	0.0183	0.0166	0.0150	0.0136	0.0123	0.0111	0.0101	0.0091
1	0.1074	0.0992	0.0916	0.0845	0.0780	0.0719	0.0663	0.0611	0.0563	0.0518
2	0.2689	0.2531	0.2381	0.2238	0.2102	0.1974	0.1851	0.1736	0.1626	0.1523
3	0.4735	0.4532	0.4335	0.4142	0.3954	0.3772	0.3594	0.3423	0.3257	0.3097
4	0.6678	0.6484	0.6288	0.6093	0.5898	0.5704	0.5512	0.5321	0.5132	0.4946
5	0.8156	0.8006	0.7851	0.7693	0.7531	0.7367	0.7199	0.7029	0.6858	0.6684
6	0.9091	0.8995	0.8893	0.8786	0.8675	0.8558	0.8436	0.8311	0.8180	0.8046
7	0.9599	0.9546	0.9489	0.9427	0.9361	0.9290	0.9214	0.9134	0.9049	0.8960
8	0.9840	0.9815	0.9786	0.9755	0.9721	0.9683	0.9642	0.9597	0.9549	0.9497
9	0.9942	0.9931	0.9919	0.9905	0.9889	0.9871	0.9851	0.9829	0.9805	0.9778
10	0.9981	0.9977	0.9972	0.9966	0.9959	0.9952	0.9943	0.9933	0.9922	0.9910
11	0.9994	0.9993	0.9991	0.9989	0.9986	0.9983	0.9980	0.9976	0.9971	0.9966
12	0.9998	0.9998	0.9997	0.9997	0.9996	0.9995	0.9993	0.9992	0.9990	0.9988
13	1.0000	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996
14	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999

$\lambda \rightarrow$ / $x \downarrow$	4.8	4.9	5.0	5.1	5.2	5.3	5.4	5.5	5.6	5.7
0	0.0082	0.0074	0.0067	0.0061	0.0055	0.0050	0.0045	0.0041	0.0037	0.0033
1	0.0477	0.0439	0.0404	0.0372	0.0342	0.0314	0.0289	0.0266	0.0244	0.0224
2	0.1425	0.1333	0.1247	0.1165	0.1088	0.1016	0.0948	0.0884	0.0824	0.0768
3	0.2942	0.2793	0.2650	0.2513	0.2381	0.2254	0.2133	0.2017	0.1906	0.1800
4	0.4763	0.4582	0.4405	0.4231	0.4061	0.3895	0.3733	0.3575	0.3422	0.3272
5	0.6510	0.6335	0.6160	0.5984	0.5809	0.5635	0.5461	0.5289	0.5119	0.4950
6	0.7908	0.7767	0.7622	0.7474	0.7324	0.7171	0.7017	0.6860	0.6703	0.6544
7	0.8867	0.8769	0.8666	0.8560	0.8449	0.8335	0.8217	0.8095	0.7970	0.7841
8	0.9442	0.9382	0.9319	0.9252	0.9181	0.9106	0.9027	0.8944	0.8857	0.8766
9	0.9749	0.9717	0.9682	0.9644	0.9603	0.9559	0.9512	0.9462	0.9409	0.9352
10	0.9896	0.9880	0.9863	0.9844	0.9823	0.9800	0.9775	0.9747	0.9718	0.9686
11	0.9960	0.9953	0.9945	0.9937	0.9927	0.9916	0.9904	0.9890	0.9875	0.9859
12	0.9986	0.9983	0.9980	0.9976	0.9972	0.9967	0.9962	0.9955	0.9949	0.9941
13	0.9995	0.9994	0.9993	0.9992	0.9990	0.9988	0.9986	0.9983	0.9980	0.9977
14	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9995	0.9994	0.9993	0.9991
15	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9998	0.9997
16	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999

$\lambda \rightarrow$ $x \downarrow$	5.8	5.9	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7
0	0.0030	0.0027	0.0025	0.0022	0.0020	0.0018	0.0017	0.0015	0.0014	0.0012
1	0.0206	0.0189	0.0174	0.0159	0.0146	0.0134	0.0123	0.0113	0.0103	0.0095
2	0.0715	0.0666	0.0620	0.0577	0.0536	0.0498	0.0463	0.0430	0.0400	0.0371
3	0.1700	0.1604	0.1512	0.1425	0.1342	0.1264	0.1189	0.1118	0.1052	0.0988
4	0.3127	0.2987	0.2851	0.2719	0.2592	0.2469	0.2351	0.2237	0.2127	0.2022
5	0.4783	0.4619	0.4457	0.4298	0.4141	0.3988	0.3837	0.3690	0.3547	0.3406
6	0.6384	0.6224	0.6063	0.5902	0.5742	0.5582	0.5423	0.5265	0.5108	0.4953
7	0.7710	0.7576	0.7440	0.7301	0.7160	0.7017	0.6873	0.6728	0.6581	0.6433
8	0.8672	0.8574	0.8472	0.8367	0.8259	0.8148	0.8033	0.7916	0.7796	0.7673
9	0.9292	0.9228	0.9161	0.9090	0.9016	0.8939	0.8858	0.8774	0.8686	0.8596
10	0.9651	0.9614	0.9574	0.9531	0.9486	0.9437	0.9386	0.9332	0.9274	0.9214
11	0.9841	0.9821	0.9799	0.9776	0.9750	0.9723	0.9693	0.9661	0.9627	0.9591
12	0.9932	0.9922	0.9912	0.9900	0.9887	0.9873	0.9857	0.9840	0.9821	0.9801
13	0.9973	0.9969	0.9964	0.9958	0.9952	0.9945	0.9937	0.9929	0.9920	0.9909
14	0.9990	0.9988	0.9986	0.9984	0.9981	0.9978	0.9974	0.9970	0.9966	0.9961
15	0.9996	0.9996	0.9995	0.9994	0.9993	0.9992	0.9990	0.9988	0.9986	0.9984
16	0.9999	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996	0.9996	0.9995	0.9994
17	1.0000	1.0000	0.9999	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998	0.9998

$\lambda \rightarrow$ $x \downarrow$	7.0	7.5	8.0	8.5	9.0	9.5	10.0	10.5	11.0	11.5
0	0.0009	0.0006	0.0003	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000	0.0000
1	0.0073	0.0047	0.0030	0.0019	0.0012	0.0008	0.0005	0.0003	0.0002	0.0001
2	0.0296	0.0203	0.0138	0.0093	0.0062	0.0042	0.0028	0.0018	0.0012	0.0008
3	0.0818	0.0591	0.0424	0.0301	0.0212	0.0149	0.0103	0.0071	0.0049	0.0034
4	0.1730	0.1321	0.0996	0.0744	0.0550	0.0403	0.0293	0.0211	0.0151	0.0107
5	0.3007	0.2414	0.1912	0.1496	0.1157	0.0885	0.0671	0.0504	0.0375	0.0277
6	0.4497	0.3782	0.3134	0.2562	0.2068	0.1649	0.1301	0.1016	0.0786	0.0603
7	0.5987	0.5246	0.4530	0.3856	0.3239	0.2687	0.2202	0.1785	0.1432	0.1137
8	0.7291	0.6620	0.5925	0.5231	0.4557	0.3918	0.3328	0.2794	0.2320	0.1906
9	0.8305	0.7764	0.7166	0.6530	0.5874	0.5218	0.4579	0.3971	0.3405	0.2888
10	0.9015	0.8622	0.8159	0.7634	0.7060	0.6453	0.5830	0.5207	0.4599	0.4017
11	0.9467	0.9208	0.8881	0.8487	0.8030	0.7520	0.6968	0.6387	0.5793	0.5198
12	0.9730	0.9573	0.9362	0.9091	0.8758	0.8364	0.7916	0.7420	0.6887	0.6329
13	0.9872	0.9784	0.9658	0.9486	0.9261	0.8981	0.8645	0.8253	0.7813	0.7330
14	0.9943	0.9897	0.9827	0.9726	0.9585	0.9400	0.9165	0.8879	0.8540	0.8153
15	0.9976	0.9954	0.9918	0.9862	0.9780	0.9665	0.9513	0.9317	0.9074	0.8783
16	0.9990	0.9980	0.9963	0.9934	0.9889	0.9823	0.9730	0.9604	0.9441	0.9236
17	0.9996	0.9992	0.9984	0.9970	0.9947	0.9911	0.9857	0.9781	0.9678	0.9542
18	0.9999	0.9997	0.9993	0.9987	0.9976	0.9957	0.9928	0.9885	0.9823	0.9738
19	1.0000	0.9999	0.9997	0.9995	0.9989	0.9980	0.9965	0.9942	0.9907	0.9857
20	1.0000	1.0000	0.9999	0.9998	0.9996	0.9991	0.9984	0.9972	0.9953	0.9925

$\lambda \rightarrow$ $x \downarrow$	12.0	12.5	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5
3	0.0023	0.0016	0.0011	0.0007	0.0005	0.0003	0.0002	0.0001	0.0001	0.0001
4	0.0076	0.0053	0.0037	0.0026	0.0018	0.0012	0.0009	0.0006	0.0004	0.0003
5	0.0203	0.0148	0.0107	0.0077	0.0055	0.0039	0.0028	0.0020	0.0014	0.0010
6	0.0458	0.0346	0.0259	0.0193	0.0142	0.0105	0.0076	0.0055	0.0040	0.0029
7	0.0895	0.0698	0.0540	0.0415	0.0316	0.0239	0.0180	0.0135	0.0100	0.0074
8	0.1550	0.1249	0.0998	0.0790	0.0621	0.0484	0.0374	0.0288	0.0220	0.0167
9	0.2424	0.2014	0.1658	0.1353	0.1094	0.0878	0.0699	0.0552	0.0433	0.0337
10	0.3472	0.2971	0.2517	0.2112	0.1757	0.1449	0.1185	0.0961	0.0774	0.0619
11	0.4616	0.4058	0.3532	0.3045	0.2600	0.2201	0.1848	0.1538	0.1270	0.1041
12	0.5760	0.5190	0.4631	0.4093	0.3585	0.3111	0.2676	0.2283	0.1931	0.1621
13	0.6815	0.6278	0.5730	0.5182	0.4644	0.4125	0.3632	0.3171	0.2745	0.2357
14	0.7720	0.7250	0.6751	0.6233	0.5704	0.5176	0.4657	0.4154	0.3675	0.3225
15	0.8444	0.8060	0.7636	0.7178	0.6694	0.6192	0.5681	0.5170	0.4667	0.4180
16	0.8987	0.8693	0.8355	0.7975	0.7559	0.7112	0.6641	0.6154	0.5660	0.5165
17	0.9370	0.9158	0.8905	0.8609	0.8272	0.7897	0.7489	0.7052	0.6593	0.6120
18	0.9626	0.9481	0.9302	0.9084	0.8826	0.8530	0.8195	0.7825	0.7423	0.6996
19	0.9787	0.9694	0.9573	0.9421	0.9235	0.9012	0.8752	0.8455	0.8122	0.7757
20	0.9884	0.9827	0.9750	0.9649	0.9521	0.9362	0.9170	0.8944	0.8682	0.8385
21	0.9939	0.9906	0.9859	0.9796	0.9712	0.9604	0.9469	0.9304	0.9108	0.8878
22	0.9970	0.9951	0.9924	0.9885	0.9833	0.9763	0.9673	0.9558	0.9418	0.9248
23	0.9985	0.9975	0.9960	0.9938	0.9907	0.9863	0.9805	0.9730	0.9633	0.9513
24	0.9993	0.9988	0.9980	0.9968	0.9950	0.9924	0.9888	0.9840	0.9777	0.9696
25	0.9997	0.9994	0.9990	0.9984	0.9974	0.9959	0.9938	0.9909	0.9869	0.9816
26	0.9999	0.9997	0.9995	0.9992	0.9987	0.9979	0.9967	0.9950	0.9925	0.9892
27	0.9999	0.9999	0.9998	0.9996	0.9994	0.9989	0.9983	0.9973	0.9959	0.9939
28	1.0000	1.0000	0.9999	0.9998	0.9997	0.9995	0.9991	0.9986	0.9978	0.9967

$\lambda \rightarrow$ $x \downarrow$	17.0	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5
5	0.0007	0.0005	0.0003	0.0002	0.0002	0.0001	0.0001	0.0000	0.0000	0.0000
6	0.0021	0.0015	0.0010	0.0007	0.0005	0.0004	0.0003	0.0002	0.0001	0.0001
7	0.0054	0.0040	0.0029	0.0021	0.0015	0.0011	0.0008	0.0006	0.0004	0.0003
8	0.0126	0.0095	0.0071	0.0052	0.0039	0.0028	0.0021	0.0015	0.0011	0.0008
9	0.0261	0.0201	0.0154	0.0117	0.0089	0.0067	0.0050	0.0037	0.0028	0.0020
10	0.0491	0.0387	0.0304	0.0237	0.0183	0.0141	0.0108	0.0082	0.0063	0.0047
11	0.0847	0.0684	0.0549	0.0438	0.0347	0.0273	0.0214	0.0167	0.0129	0.0099
12	0.1350	0.1116	0.0917	0.0748	0.0606	0.0488	0.0390	0.0310	0.0245	0.0193
13	0.2009	0.1699	0.1426	0.1189	0.0984	0.0809	0.0661	0.0537	0.0434	0.0348
14	0.2808	0.2426	0.2081	0.1771	0.1497	0.1257	0.1049	0.0869	0.0716	0.0586
15	0.3715	0.3275	0.2867	0.2490	0.2148	0.1840	0.1565	0.1323	0.1111	0.0927
16	0.4677	0.4204	0.3751	0.3321	0.2920	0.2550	0.2211	0.1904	0.1629	0.1385
17	0.5640	0.5160	0.4686	0.4226	0.3784	0.3364	0.2970	0.2605	0.2270	0.1965
18	0.6550	0.6089	0.5622	0.5156	0.4695	0.4246	0.3814	0.3403	0.3017	0.2657

$\lambda \rightarrow$ $x \downarrow$	17.0	17.5	18.0	18.5	19.0	19.5	20.0	20.5	21.0	21.5
19	0.7363	0.6945	0.6509	0.6061	0.5606	0.5151	0.4703	0.4265	0.3843	0.3440
20	0.8055	0.7694	0.7307	0.6898	0.6472	0.6034	0.5591	0.5148	0.4710	0.4282
21	0.8615	0.8319	0.7991	0.7636	0.7255	0.6854	0.6437	0.6010	0.5577	0.5144
22	0.9047	0.8815	0.8551	0.8256	0.7931	0.7580	0.7206	0.6813	0.6405	0.5987
23	0.9367	0.9193	0.8989	0.8755	0.8490	0.8196	0.7875	0.7528	0.7160	0.6774
24	0.9594	0.9468	0.9317	0.9139	0.8933	0.8697	0.8432	0.8140	0.7822	0.7480
25	0.9748	0.9661	0.9554	0.9424	0.9269	0.9087	0.8878	0.8641	0.8377	0.8086
26	0.9848	0.9791	0.9718	0.9626	0.9514	0.9380	0.9221	0.9037	0.8826	0.8588
27	0.9912	0.9875	0.9827	0.9765	0.9687	0.9591	0.9475	0.9337	0.9175	0.8988
28	0.9950	0.9928	0.9897	0.9857	0.9805	0.9739	0.9657	0.9557	0.9436	0.9294
29	0.9973	0.9959	0.9941	0.9915	0.9882	0.9838	0.9782	0.9712	0.9626	0.9522
30	0.9986	0.9978	0.9967	0.9951	0.9930	0.9902	0.9865	0.9818	0.9758	0.9685
31	0.9993	0.9988	0.9982	0.9973	0.9960	0.9943	0.9919	0.9888	0.9848	0.9798
32	0.9996	0.9994	0.9990	0.9985	0.9978	0.9967	0.9953	0.9933	0.9907	0.9874
33	0.9998	0.9997	0.9995	0.9992	0.9988	0.9982	0.9973	0.9961	0.9945	0.9923
34	0.9999	0.9999	0.9998	0.9996	0.9994	0.9990	0.9985	0.9978	0.9968	0.9954
35	1.0000	0.9999	0.9999	0.9998	0.9997	0.9995	0.9992	0.9988	0.9982	0.9974
36	1.0000	1.0000	0.9999	0.9999	0.9998	0.9997	0.9996	0.9993	0.9990	0.9985

$\lambda \rightarrow$ $x \downarrow$	22.0	22.5	23.0	23.5	24.0	24.5	25.0	25.5	26.0	26.5
8	0.0006	0.0004	0.0003	0.0002	0.0002	0.0001	0.0001	0.0001	0.0000	0.0000
9	0.0015	0.0011	0.0008	0.0006	0.0004	0.0003	0.0002	0.0002	0.0001	0.0001
10	0.0035	0.0027	0.0020	0.0015	0.0011	0.0008	0.0006	0.0004	0.0003	0.0002
11	0.0076	0.0058	0.0044	0.0033	0.0025	0.0019	0.0014	0.0011	0.0008	0.0006
12	0.0151	0.0118	0.0091	0.0070	0.0054	0.0041	0.0031	0.0024	0.0018	0.0014
13	0.0278	0.0221	0.0174	0.0137	0.0107	0.0083	0.0065	0.0050	0.0038	0.0029
14	0.0477	0.0386	0.0311	0.0249	0.0198	0.0157	0.0124	0.0097	0.0076	0.0059
15	0.0769	0.0634	0.0520	0.0424	0.0344	0.0278	0.0223	0.0178	0.0142	0.0112
16	0.1170	0.0983	0.0821	0.0681	0.0563	0.0462	0.0377	0.0307	0.0248	0.0200
17	0.1690	0.1445	0.1228	0.1037	0.0871	0.0728	0.0605	0.0500	0.0411	0.0336
18	0.2325	0.2022	0.1748	0.1502	0.1283	0.1090	0.0920	0.0773	0.0646	0.0537
19	0.3060	0.2705	0.2377	0.2076	0.1803	0.1556	0.1336	0.1140	0.0968	0.0818
20	0.3869	0.3474	0.3101	0.2751	0.2426	0.2128	0.1855	0.1608	0.1387	0.1189
21	0.4716	0.4298	0.3894	0.3507	0.3139	0.2794	0.2473	0.2176	0.1905	0.1658
22	0.5564	0.5141	0.4723	0.4313	0.3917	0.3537	0.3175	0.2835	0.2517	0.2223
23	0.6374	0.5965	0.5551	0.5138	0.4728	0.4328	0.3939	0.3565	0.3209	0.2874
24	0.7117	0.6738	0.6346	0.5945	0.5540	0.5135	0.4734	0.4341	0.3959	0.3592
25	0.7771	0.7433	0.7077	0.6704	0.6319	0.5926	0.5529	0.5132	0.4739	0.4354
26	0.8324	0.8035	0.7723	0.7390	0.7038	0.6672	0.6294	0.5908	0.5519	0.5130
27	0.8775	0.8537	0.8274	0.7987	0.7677	0.7348	0.7002	0.6641	0.6270	0.5892
28	0.9129	0.8940	0.8726	0.8488	0.8225	0.7940	0.7634	0.7309	0.6967	0.6613
29	0.9398	0.9253	0.9085	0.8894	0.8679	0.8440	0.8179	0.7896	0.7593	0.7271

$\lambda \rightarrow$ $x \downarrow$	22.0	22.5	23.0	23.5	24.0	24.5	25.0	25.5	26.0	26.5
30	0.9595	0.9487	0.9360	0.9212	0.9042	0.8849	0.8633	0.8395	0.8134	0.7853
31	0.9735	0.9657	0.9564	0.9453	0.9322	0.9172	0.8999	0.8805	0.8589	0.8351
32	0.9831	0.9777	0.9711	0.9630	0.9533	0.9419	0.9285	0.9132	0.8958	0.8763
33	0.9895	0.9859	0.9813	0.9756	0.9686	0.9602	0.9502	0.9385	0.9249	0.9094
34	0.9936	0.9913	0.9882	0.9843	0.9794	0.9734	0.9662	0.9574	0.9472	0.9352
35	0.9962	0.9947	0.9927	0.9902	0.9868	0.9827	0.9775	0.9713	0.9637	0.9547
36	0.9978	0.9969	0.9956	0.9940	0.9918	0.9890	0.9854	0.9810	0.9756	0.9691
37	0.9988	0.9982	0.9974	0.9964	0.9950	0.9932	0.9908	0.9878	0.9840	0.9793
38	0.9993	0.9990	0.9985	0.9979	0.9970	0.9958	0.9943	0.9923	0.9897	0.9865
39	0.9996	0.9995	0.9992	0.9988	0.9983	0.9975	0.9966	0.9953	0.9936	0.9914
40	0.9998	0.9997	0.9996	0.9993	0.9990	0.9986	0.9980	0.9971	0.9961	0.9946
41	0.9999	0.9998	0.9998	0.9996	0.9995	0.9992	0.9988	0.9983	0.9976	0.9967
42	1.0000	0.9999	0.9999	0.9998	0.9997	0.9995	0.9993	0.9990	0.9986	0.9980